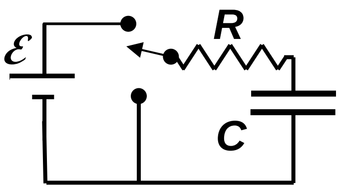
Electricity and Magnetism
Lecture 13 - Physics 121
Electromagnetic Oscillations in LC & LCR Circuits,
Y&F Chapter 30, Sec. 5 - 6
Alternating Current Circuits, Y&F Chapter 31, Sec. 1 - 2

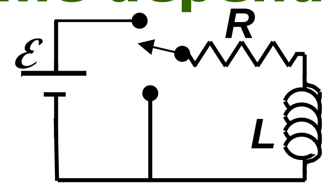
- **Summary: RC and LC circuits**
- **Mechanical Harmonic Oscillator**
- **LC Circuit Oscillations**
- **Damped Oscillations in an LCR Circuit**
- **AC Circuits, Phasors, Forced Oscillations**
- **Phase Relations for Current and Voltage in Simple Resistive, Capacitive, Inductive Circuits.**
- **Summary**

Recap: LC and RC circuits with constant EMF - – Time dependent effects



$$V_c(t) = \frac{Q(t)}{C}$$

$\tau_C \equiv RC \equiv$ capacitive time constant



$$\mathcal{E}_L(t) = -L\frac{di}{dt}$$

$\tau_L \equiv L/R \equiv$ inductive time constant

Growth Phase

$$Q(t) = Q_{inf}\left(1 - e^{-t/RC}\right) \quad Q_{inf} \equiv C\mathcal{E}$$

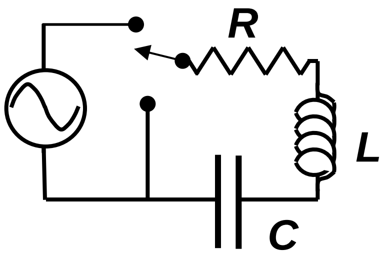
$$i(t) = i_{inf}\left(1 - e^{-t/\tau_L}\right) \quad i_{inf} \equiv \frac{\mathcal{E}}{R}$$

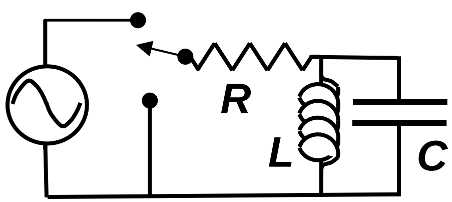
Decay Phase

$$Q(t) = Q_0 e^{-t/RC} \quad Q_0 \equiv C\mathcal{E}$$

$$i(t) = i_0 e^{-t/\tau_L} \quad i_0 \equiv \frac{\mathcal{E}}{R}$$

Now LCR in same circuit, time varying EMF –> New effects





- **Generalized Resistances: Reactances, Impedance**

  $R \qquad X_C \equiv 1/(\omega C) \qquad X_L \equiv \omega L \qquad$ [Ohms]

  $$Z \equiv \left[R^2 + (X_L - X_C)^2\right]^{1/2}$$

- **New behavior: Resonant Oscillation in LC Circuit**

  $$\omega_{res} = (LC)^{-1/2}$$

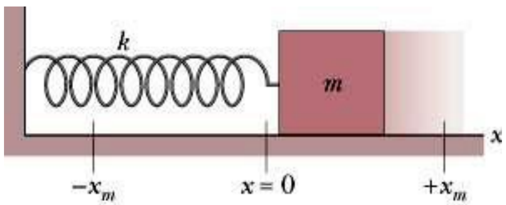
- **New behavior: Damped Oscillation in LCR Circuit**
- **External AC can drive circuit, frequency $\omega_D = 2\pi f$**

Recall: Resonant mechanical oscillations

Definition of an oscillating system:

- Periodic, repetitive behavior
- System state (t) = state(t + T) = ...= state(t + NT)
- T = period = time to complete one complete cycle
- State can mean: position and velocity, electric and magnetic fields,...

Mechanical example: Spring oscillator (simple harmonic motion)



no friction

Hooke's Law restoring force: $F = -kx = ma$ *OR*

$$E_{mech} = \text{constant} = K_{block} + U_{sp} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\frac{dE_{mech}}{dt} = 0 = m\frac{d^2x}{dt^2}\frac{dx}{dt} + kx\frac{dx}{dt} \qquad v \equiv \frac{dx}{dt} \neq 0$$

Systems that oscillate obey equations like this...	$\frac{d^2x}{dt^2} = -\omega_0^2 x \qquad \omega_0 \equiv \sqrt{k/m}$
With solutions like this...	$x(t) = x_0 \cos(\omega_0 t + \phi)$

What oscillates for a spring in SHM? position & velocity, spring force,

Energy oscillates between 100% kinetic and 100% potential: $K_{max} = U_{max}$

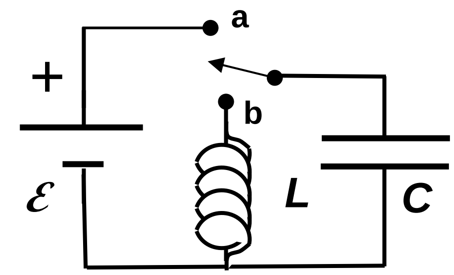
Substitutions can convert mechanical to LC equations:

$$x \to Q \quad v \to i \quad m \to L \quad k \to 1/C \qquad \omega_0^2 = k/m \;\to\; \omega_0^2 = 1/LC$$

$$U_{el} = \tfrac{1}{2}kx^2 \;\to\; U_C = \tfrac{1}{2}\frac{Q^2}{C} \qquad K_{block} = \tfrac{1}{2}mv^2 \to U_L = \tfrac{1}{2}Li^2$$

Electrical Oscillations in an LC circuit, zero resistance

Charge capacitor fully to $Q_0 = C\mathcal{E}$ then switch to "b"



Kirchoff loop equation: $V_C - \mathcal{E}_L = 0$

Substitute: $$V_C = \frac{Q}{C} \quad \text{and} \quad \mathcal{E}_L = -L\frac{di}{dt} = -L\frac{d^2Q}{dt^2}$$

$$\frac{d^2Q(t)}{dt^2} = -\frac{1}{LC}Q(t)$$

An oscillator equation where $\sqrt{1/LC} \equiv \omega_0$

solution: $Q(t) = Q_0\cos(\omega_0 t + \phi) \qquad Q_0 \equiv C\mathcal{E}$

$$i(t) = \frac{dQ(t)}{dt} = i_0\sin(\omega_0 t + \phi) \qquad i_0 \equiv \omega_0 Q_0$$

What oscillates? Charge, current, B & E fields, U_B, U_E

Peaks of current and charge are out of phase by 90^0

$$\omega_0^2 = \frac{1}{\tau_C \tau_L}$$

where $\tau_C = RC, \tau_L = L/R$

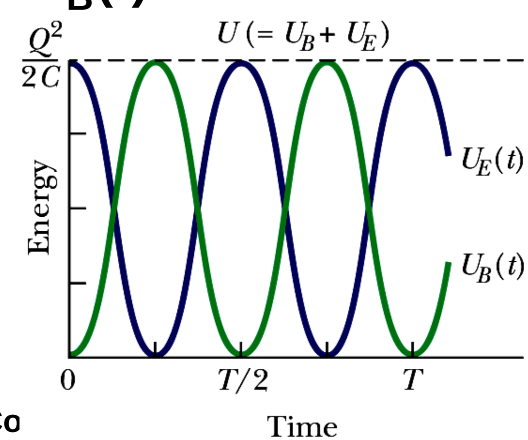
Show: Total potential energy is constant $U_{tot} = U_E(t) + U_B(t)$

$$U_E = \frac{Q^2(t)}{2C} = \frac{Q_0^2}{2C}\cos^2(\omega_0 t + \phi) \qquad U_B = \frac{Li^2(t)}{2} = \frac{Li_0^2}{2}\sin^2(\omega_0 t + \phi)$$

Peak Values Are Equal

$$U_{B_0} \equiv \frac{Li_0^2}{2} = \frac{L\omega_0^2 Q_0^2}{2} = \frac{LQ_0^2}{2}\frac{1}{LC} = \frac{Q_0^2}{2C} \equiv U_{E_0}$$

$U_{E_0} = U_{B_0}$ → U_{tot} is constant



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Details: use energy conservation to deduce oscillations

- **The total energy:** $U = U_B + U_E = \frac{1}{2}Li(t)^2 + \frac{Q(t)^2}{2C}$
- **It is constant so:** $\frac{dU}{dt} = 0 = Li\frac{di}{dt} + \frac{Q}{C}\frac{dQ}{dt}$
- **The definitions** $i \equiv \frac{dQ}{dt}$ **and** $\frac{di}{dt} = \frac{d^2Q}{dt^2}$ **imply that:** $\frac{dQ}{dt}\left(L\frac{d^2Q}{dt^2} + \frac{Q}{C}\right) = 0$
- **Either** $\frac{dQ}{dt} = 0$ **(no current ever flows) or:** $\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$ oscillator equation
- **Oscillator solution:** $Q = Q_0\cos(\omega_0 t + \phi)$

- **To evaluate ω_0: plug the first and second derivatives of the solution into the differential equation.**

$$\frac{dQ}{dt} = -Q_0\omega_0\sin(\omega_0 t + \phi)$$
$$\frac{d^2Q}{dt^2} = -Q_0\omega_0^2\cos(\omega_0 t + \phi)$$

$$\Rightarrow \quad \frac{d^2Q}{dt^2} + \frac{Q}{LC} = Q_0\cos(\omega_0 t + \phi)\left[-\omega_0^2 + \frac{1}{LC}\right] = 0$$

- **The resonant oscillation frequency ω_0 is:** $\omega_0 = \frac{1}{\sqrt{LC}}$

Oscillations Forever?

13 – 1: What do you think will happen to the oscillations in a true LC circuit (versus a real circuit) over a long time?

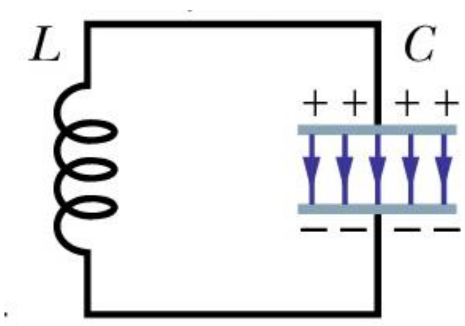
A. They will stop after one complete cycle.

B. They will continue forever.

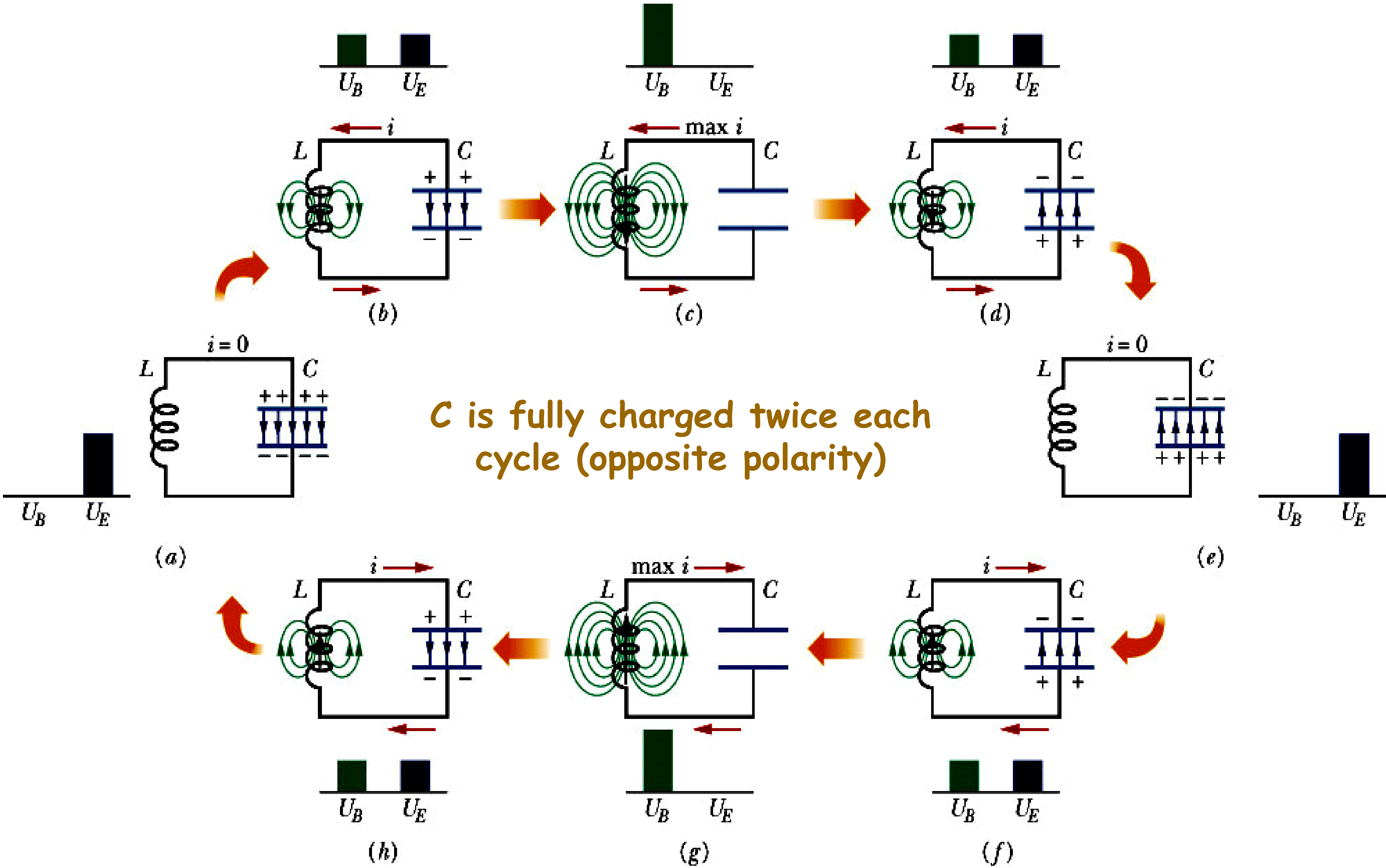
C. They will continue for awhile, and then suddenly stop.

D. They will continue for awhile, but eventually die away.

E. There is not enough information to tell what will happen.

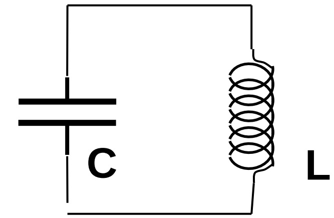


Potential Energy alternates between all electrostatic and all magnetic – two reversals per period τ



Example:

A 4 μF capacitor is charged to $\mathcal{E}$ = 5.0 V, and then discharged through a 0.3 Henry inductance in an LC circuit



Use preceding solutions with $\phi = 0$

a) Find *the oscillation period and frequency* $f = \omega / 2\pi$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.3)(4 \times 10^{-6})}} = 913 \text{ rad/s}$$

$$f = \omega_0 / 2\pi = 145 \text{ Hz}$$

$$T = \text{Period} = \frac{1}{f} = \frac{2\pi}{\omega_0} = 6.9 \text{ ms}$$

b) Find the maximum (peak) current (differentiate Q(t))

$$Q(t) = Q_0 \cos(\omega_0 t)$$

$$Q_0 = C\mathcal{E}$$

$$i(t) = \frac{dQ}{dt} = Q_0 \frac{d}{dt}\cos(\omega_0 t) = -Q_0\omega_0 \sin_0(\omega_0 t)$$

$$i_{max} = Q_0\omega_0 = C\mathcal{E}\omega_0 = 4\text{x}10^{-6} \text{ x } 5 \text{ x } 913 = 18 \text{ mA}$$

c) When does the first current maximum occur? When $|\sin(\omega_0 t)| = 1$

Maxima of Q(t): All energy is in E field
Maxima of i(t): All energy is in B field
Current maxima at T/4, 3T/4, … (2n+1)T/4

First One:

$$T = \frac{1}{f} = 6.9 \text{ ms} \Rightarrow \frac{T}{4} = 1.7 \text{ ms}$$

Others at:

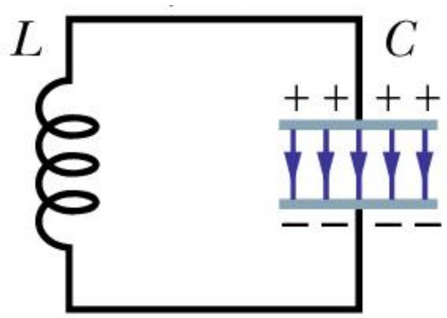
3.4 ms increments

Example

a) Find the voltage across the capacitor in the circuit as a function of time.

$L = 30\ \text{mH}$, $C = 100\ \mu\text{F}$

The capacitor is charged to $Q_0 = 0.001$ Coul. at time $t = 0$.



The resonant frequency is:

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(3\times10^{-2}\text{H})(10^{-4}\text{F})}} = 577.4\ \text{rad/s}$$

The voltage across the capacitor has the same time dependence as the charge:

$$V_C(t) = \frac{Q(t)}{C} = \frac{Q_0\cos(\omega_0 t + \phi)}{C}$$

At time $t = 0$, $Q = Q_0$, so choose phase angle $\phi = 0$.

$$V_C(t) = \frac{10^{-3}\text{C}}{10^{-4}\text{F}}\cos(577t) = 10\cos(577t) \quad \text{volts}$$

b) What is the expression for the current in the circuit? The current is:

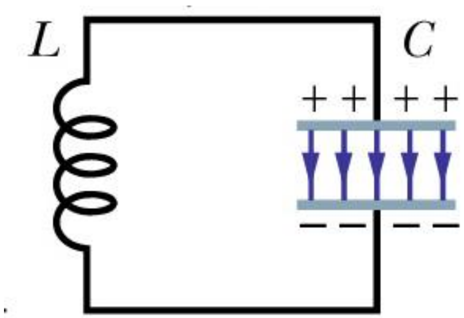
$$i = \frac{dQ}{dt} = -Q_0\omega_0\sin(\omega_0 t) = -(10^{-3}\text{C})(577\ \text{rad/s})\sin(577t) = -0.577\sin(577t)\ \text{amps}$$

c) How long until the capacitor charge is reversed? That happens every ½ period, given by:

$$\frac{T}{2} = \frac{\pi}{\omega_0} = 5.44\ \text{ms}$$

Which Current is Greatest?

13 – 2: The expressions below could be used to represent the charge on a capacitor in an LC circuit. Which one has the greatest maximum current magnitude?



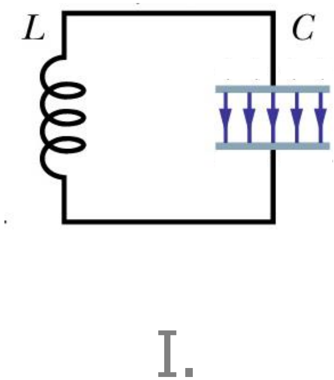
A. Q(t) = 2 sin(5t)
B. Q(t) = 2 cos(4t)
C. Q(t) = 2 cos(4t+π/2)
D. Q(t) = 2 sin(2t)
E. Q(t) = 4 cos(2t)

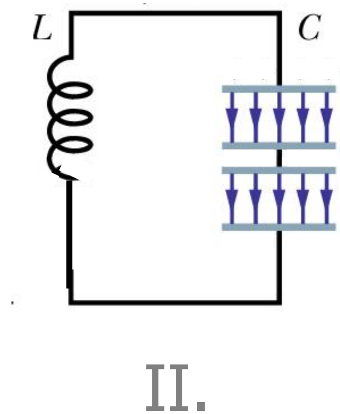
$$Q(t) = Q_0 \cos(\omega_0 t + \phi) \qquad i = \frac{dQ(t)}{dt}$$

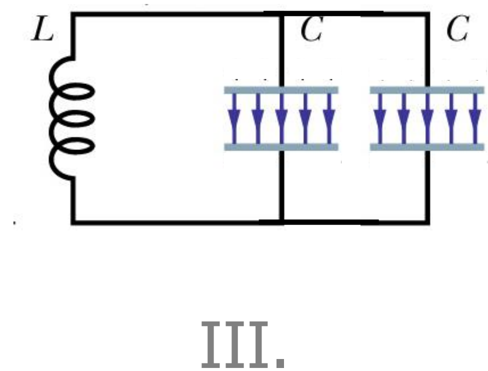
Time needed to discharge the capacitor in LC circuit

13 – 3: The three LC circuits below have identical inductors and capacitors. Order the circuits according to their oscillation frequency in ascending order.

A. I, II, III.
B. II, I, III.
C. III, I, II.
D. III, II, I.
E. II, III, I.



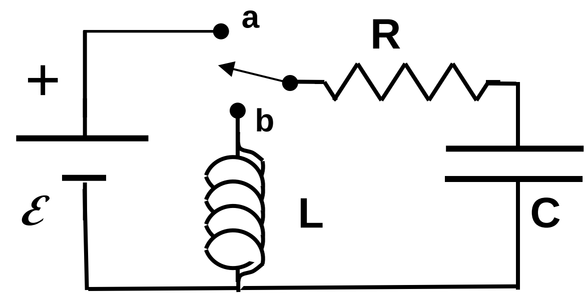




$$\omega_0 \equiv \sqrt{1/LC} = \frac{2\pi}{T} \qquad C_{para} \equiv \sum C_i \qquad 1/C_{ser} \equiv \sum 1/C_i$$

LCR circuits: Add series resistance
Circuits still oscillate but oscillation is *damped*

Charge capacitor fully to $Q_0 = C\mathcal{E}$ then switch to "b"
Stored energy decays with time due to resistance



$$U_{tot} = U_E(t) + U_B(t) = \frac{Q^2(t)}{2C} + \frac{Li^2(t)}{2}$$

Resistance dissipates stored energy. The power is:

$$\frac{dU_{tot}}{dt} = -i^2(t)R$$

Oscillator equation results, but with damping (decay) term

$$\frac{d^2Q(t)}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \omega_0^2 Q(t) = 0 \qquad \omega_0 \equiv \sqrt{1/LC}$$

Solution is cosine with exponential decay (weak or under-damped case)

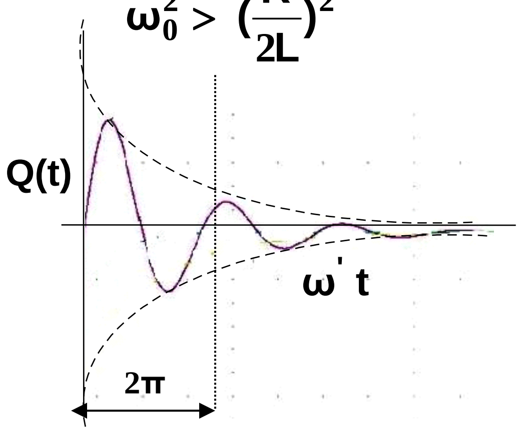
$$Q(t) = Q_0 e^{-Rt/2L}\cos(\omega' t + \phi) \qquad Q_0 \equiv C\mathcal{E}$$

Shifted resonant frequency ω' can be real or imaginary

$$\omega' \equiv \left[\omega_0^2 - (R/2L)^2\right]^{1/2}$$

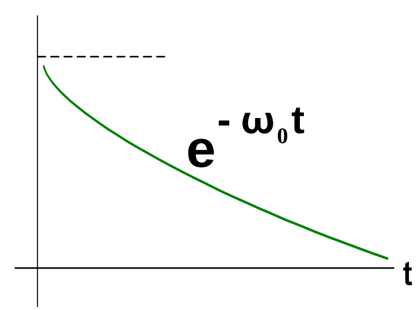
Underdamped:

$$\omega_0^2 > \left(\frac{R}{2L}\right)^2$$



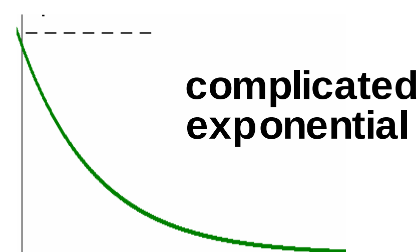
Critically damped

$$\omega' = 0 \qquad \omega_0^2 = \left(\frac{R}{2L}\right)^2$$



Overdamped

ω' is imaginary $\qquad \omega_0^2 < \left(\frac{R}{2L}\right)^2$



Resonant frequency with damping

13 – 4: How does the resonant frequency ω_0 for an ideal LC circuit (no resistance) compare with ω' for an under-damped circuit whose resistance cannot be ignored?

A. The resonant frequency for the non-ideal, damped circuit is *higher* than for the ideal one ($\omega' > \omega_0$).

B. The resonant frequency for the damped circuit is *lower* than for the ideal one ($\omega' < \omega_0$).

C. The resistance in the circuit does not affect the resonant frequency—they are the same ($\omega' = \omega_0$).

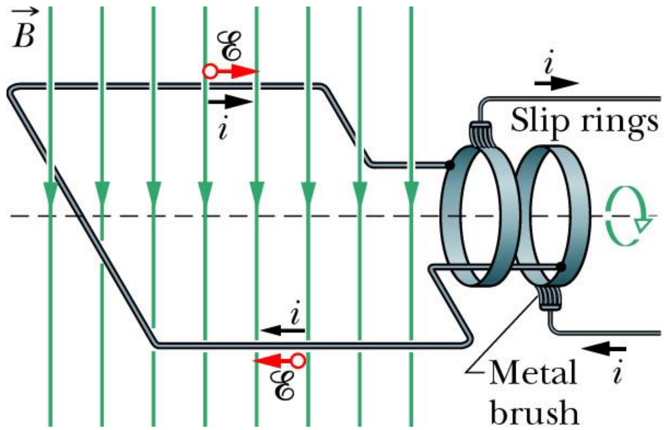
D. The damped circuit has an *imaginary* value of ω'.

$$\omega' \equiv \left[\omega_0^2 - (R/2L)^2\right]^{1/2}$$

Alternating Current (AC) - EMF

- AC is easier to transmit than DC
- AC transmission voltage can be changed by using a transformer.
- Commercial electric power (home or office) is AC, not DC.
- The U.S., the AC frequency is 60 Hz. Most other countries use 50 Hz.

- Sketch: a crude AC *generator*.
- EMF appears in a rotating a coil of wire in a magnetic field (Faraday's Law)
- Slip rings and brushes allow the EMF to be taken off the coil without twisting the wires.
- Generators convert mechanical energy to electrical energy. Power to rotate the coil can come from a water or steam turbine, windmill, or turbojet engine.



Represent outputs as sinusoidal functions:

$$\varepsilon_{ind}(t) = \varepsilon_m \sin\omega_d t$$

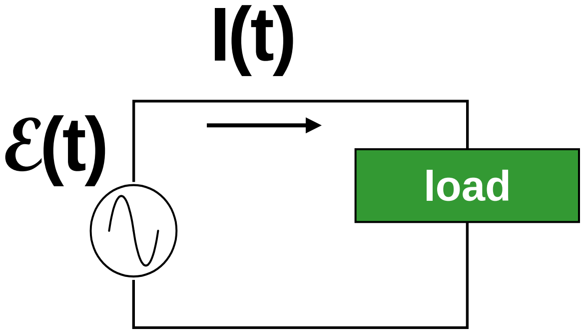
$$i = i_0 \sin(\omega_d t - \phi)$$

External AC EMF driving a circuit

External, sinusoidal, instantaneous EMF applied to load:

$$\mathcal{E}(t) = \mathcal{E}_{max}\sin(\omega_D t)$$

$\mathcal{E}_{max} \equiv$ amplitude



The potential across the load $V_{load}(t) = \mathcal{E}(t)$

ω_D = the driving frequency

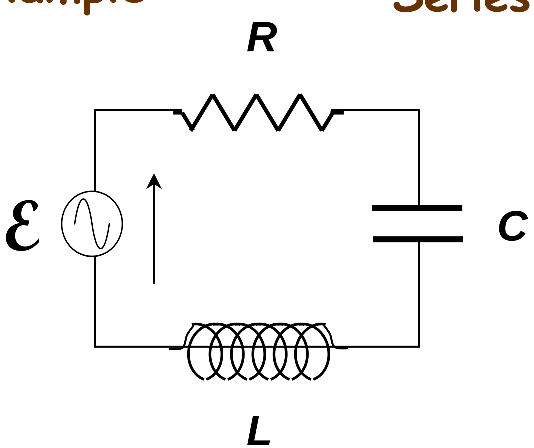
$\omega_D \neq$ resonant frequency ω_0, in general

Current in load flows with same frequency ω_D ...but may be retarded or advanced (relative to $\mathcal{E}$) by "phase angle" Φ (due to inertia L and stiffness 1/C)
Current has the same amplitude and phase everywhere in a branch

$$I(t) = I_{max}\sin(\omega_D t - \Phi)$$

Example:

Series LCR Circuit



Everything oscillates at driving frequency ω_D

At "resonance": $\omega_D = \omega_0 = \sqrt{1/LC}$

Φ is zero at resonance - circuit acts purely resistively.

Otherwise Φ is + or - (current leads or lags applied EMF)

Instantaneous, peak, average, and RMS quantities for AC circuits

$$\mathcal{E}(t) = \mathcal{E}_{max}\sin(\omega_D t) \qquad i(t) = i_{max}\sin(\omega_D t - \Phi)$$

Instantaneous voltages and currents:

- depend on time through argument: ωt
- periodic, repetitive, oscillatory
- possibly advanced or retarded relative to each other by phase angles
- represented by rotating "phasors" – see below

Peak voltage and current amplitudes are just the coefficients out front

$$\mathcal{E}_{max} \qquad i_{max}$$

Simple time averages of periodic quantities are zero (and useless).

- Example: Integrate over a **whole** number of periods – one τ is enough ($\omega t = 2\pi$)

$$\int_0^\tau \sin(\omega t - \Phi)\,dt = 0$$

Integrand is odd

$$\mathcal{E}_{av} \equiv \frac{1}{\tau}\int_0^\tau \mathcal{E}(t)\,dt = \mathcal{E}_{max}\frac{1}{\tau}\int_0^\tau \sin(\omega_D t)\,dt = 0$$

$$i_{av} \equiv \frac{1}{\tau}\int_0^\tau i(t)\,dt = i_{max}\frac{1}{\tau}\int_0^\tau \sin(\omega_D t - \Phi)\,dt = 0$$

"RMS" averages are used the way instantaneous quantities were in DC circuits

- "RMS" means "root, mean, squared".

$$\frac{1}{\tau}\int_0^\tau \sin^2(\omega t - \Phi)\,dt = 1/2$$

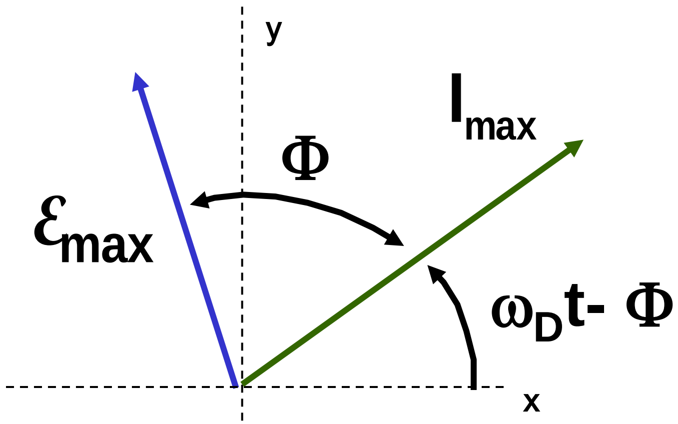
Integrand is even

$$\mathcal{E}_{rms} \equiv \left[\left\langle \mathcal{E}^2 \right\rangle_{av}\right]^{1/2} = \left[\frac{1}{\tau}\int_0^\tau \mathcal{E}^2(t)\,dt\right]^{1/2} = \frac{\mathcal{E}_{max}}{\sqrt{2}}$$

$$i_{rms} \equiv \left[\left\langle i^2(t) \right\rangle_{av}\right]^{1/2} = \left[\frac{1}{\tau}\int_0^\tau i^2(t)\,dt\right]^{1/2} = \frac{i_{max}}{\sqrt{2}}$$

Phasor Picture:

Show current and potentials as vectors rotating at frequency ω_D



$$\mathcal{E}(t) = \mathcal{E}_{max}\sin(\omega_D t)$$

$$I(t) = i_{max}\sin(\omega_D t - \Phi)$$

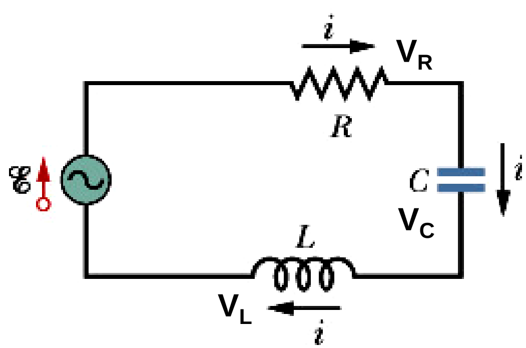
- The ***measured instantaneous*** values of i(t) and $\mathcal{E}$(t) are ***projections*** of the phasors on the y-axis.
- The lengths of the vectors are the ***peak*** amplitudes.

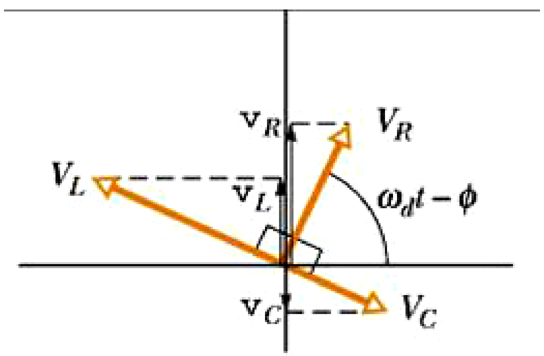
Φ ≡ "phase angle" measures when peaks pass

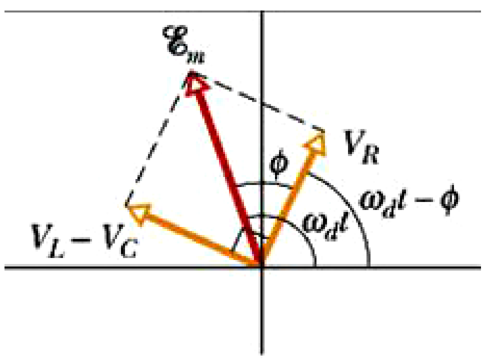
Φ and $\omega_D t$ are independent

- Current is the same (phase included) everywhere in a single branch of any circuit.
- EMF $\mathcal{E}$(t) applied to the circuit can lead or lag the current by a phase angle Φ in the range [-π/2, +π/2].

Series LCR circuit: Relate internal voltage drops to phase of the current

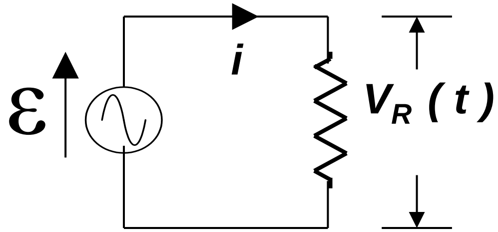






- Voltage across R is in phase with the current.
- Voltage across C lags the current by 90^0.
- Voltage across L leads the current by 90^0.

AC circuit, resistance only → current and voltage in phase



Kirchoff loop rule: $\varepsilon - V_R(t) = 0$

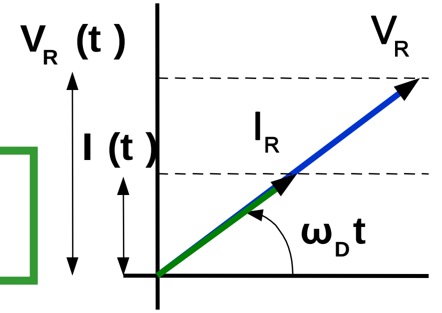
$$\Rightarrow \quad V_R(t) = \varepsilon(t) = \varepsilon_M \sin(\omega_D t)$$

Current $i_R(t) = i_{R\,max} \sin(\omega_D t - \Phi)$

Voltage drop across R: $V_R(t) \equiv i_R(t)R$ (definition of resistance)

$$\frac{\mathcal{E}_m}{i_R} \equiv R$$

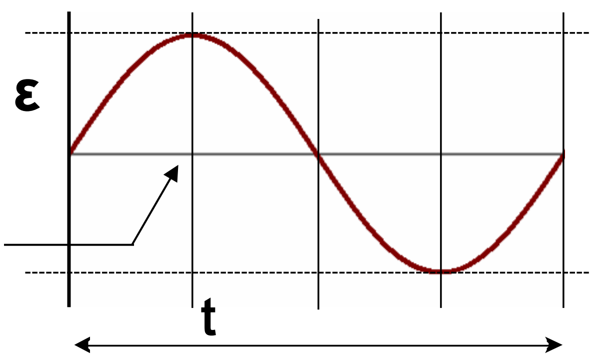
$$\therefore \; \Phi = 0$$



Peak current and peak voltage are in phase in a purely resistive part of a circuit, rotating at the driving frequency ω_D

Example

$\varepsilon_M = 10V$, $f = 60\,Hz$ applied to a $20\,\Omega$ Resistor. Find the current $i(t)$ at $t = \frac{1}{240}$ s



$t = \frac{1}{240}$

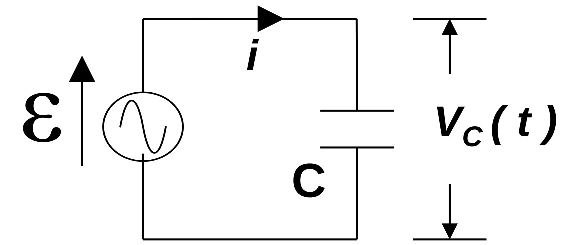
$T = \frac{1}{f} = \frac{1}{60}$

$\varepsilon_M \qquad \omega_D = 2\pi f = 120\pi \qquad \tau = 1/60$ s. $\qquad t = \tau/4$

$$i(t) = i_{Rmax} \sin(\omega_D t) = \frac{\mathcal{E}_m}{R} \sin(120\pi \,.\, 1/240)$$

$$i(t) = \frac{10}{20} \sin\frac{\pi}{2} = \frac{1}{2}\,A$$

Capacitance-only AC circuit: current leads voltage by $\pi/2$



Kirchoff loop rule: $\varepsilon(t) - V_c(t) = 0$

$\Rightarrow V_c(t) = V_{Cmax}\sin(\omega_D t); \quad V_{Cmax} \equiv \mathcal{E}_m$

Charge: $Q(t) = C\,V_C(t) = C\,V_{Cmax}\sin(\omega_D t);$

Current: $i_c(t) = \dfrac{dQ(t)}{dt} = \omega_D\,C\,V_{Cmax}\cos(\omega_D t) \equiv i_{Cmax}\sin(\omega_D t - \Phi)$

Note: $\cos(\omega_D t) = \sin(\omega_D t + \dfrac{\pi}{2})$

so...phase angle $\Phi = -\pi/2$ for capacitor

$V_{Cmax} \equiv i_{Cmax} X_C \quad (\text{recall } V_R = i_R R)$

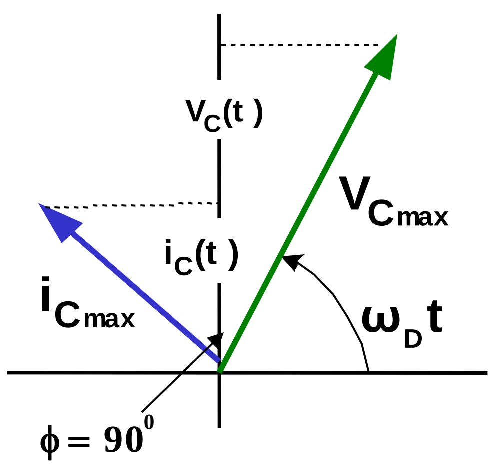
Reactances are ratios of peak values

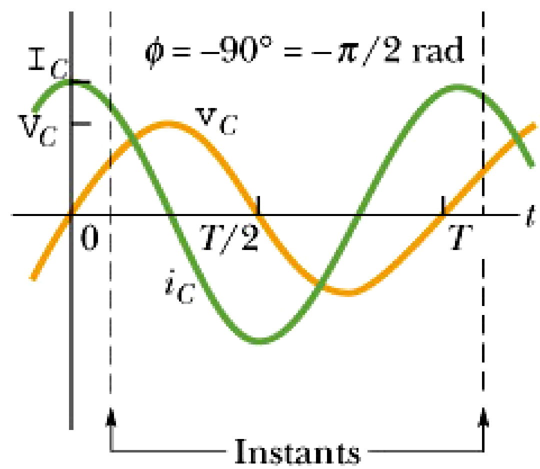
Definition: capacitive reactance $\quad X_C \equiv \dfrac{1}{\omega_D C}$ (Ohms)

Current leads the voltage by $\pi/2$ in a pure capacitive part of a circuit (Φ negative)

$\therefore\; i_c(t) = \dfrac{V_{Cmax}}{X_C}\sin(\omega_D t + \dfrac{\pi}{2})$

Phasor Picture

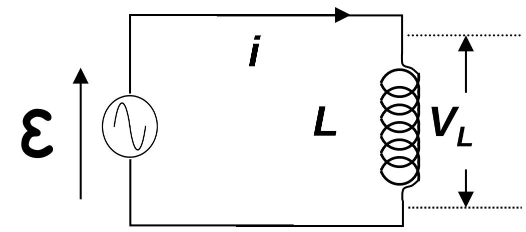




Capacitive Reactance limiting cases

- $\omega \rightarrow 0$: Infinite reactance. DC blocked. C acts like broken wire.
- $\omega \rightarrow$ infinity: Reactance is zero. Capacitor acts like a simple wire

Inductance-only AC circuit: current lags voltage by $\pi/2$



Kirchoff loop rule: $\mathcal{E}(t) + V_L(t) = 0$

$$\Rightarrow \quad V_L(t) = -\mathcal{E}_m \sin(\omega_D t); \quad \mathcal{E}_m \equiv V_{L\,max}$$

Faraday Law: $V_L(t) = -L\dfrac{di_L}{dt} \Rightarrow \dfrac{di_L}{dt} = -\dfrac{V_L(t)}{L}$

Current: $i_L(t) = \int di_L = \dfrac{V_{L\,max}}{L}\int \sin(\omega_D t)\,dt = -\dfrac{V_{L\,max}}{\omega_D L}\cos(\omega_D t) \equiv i_{L\,max}\sin(\omega_D t - \Phi)$

Note: $\cos(\omega_D t) = -\sin(\omega_D t - \dfrac{\pi}{2})$

so...phase angle $\Phi = +\pi/2$ for inductor

$$V_{L\,max} \equiv i_{L\,max} X_L$$

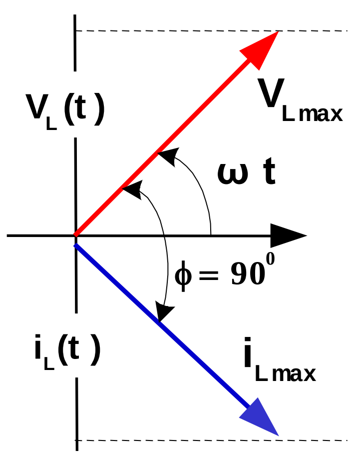
Reactances are ratios of peak values

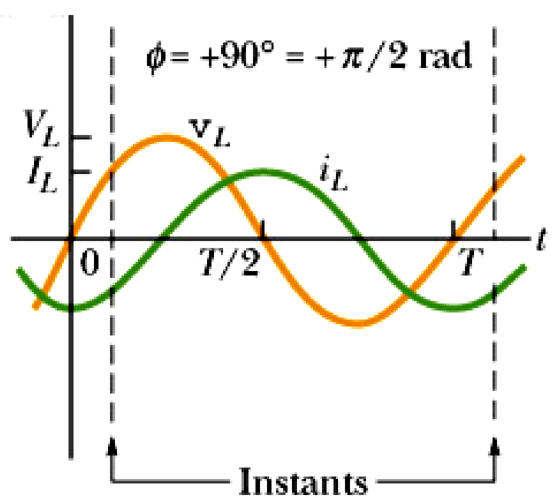
Definition: inductive reactance $X_L \equiv \omega_D L$ (Ohms)

Current lags the voltage by $\pi/2$ in a pure inductive part of a circuit (Φ positive)

$$i_{L\,max}(t) = \frac{V_{L\,max}}{X_L}\sin(\omega_D t - \frac{\pi}{2})$$

Phasor Picture





Inductive Reactance limiting cases

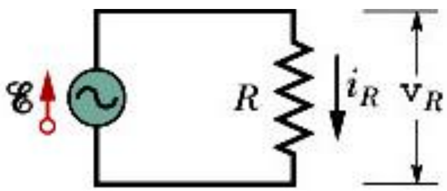
- $\omega \to 0$: Zero reactance. Inductance acts like a wire.
- $\omega \to$ infinity: Infinite reactance. Inductance acts like a broken wire.

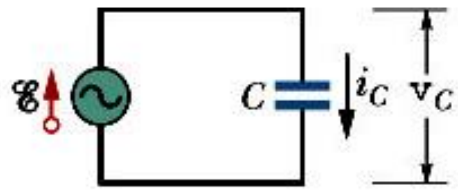
Current & voltage phases in pure R, C, and L circuits

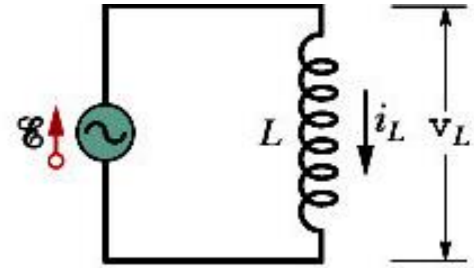
Current is the same everywhere in a single branch (including phase)

Phases of voltages in elements are referenced to the current phasor

- Apply sinusoidal voltage $\mathcal{E}(t) = \mathcal{E}_m \sin(\omega_D t)$
- For pure R, L, or C loads, phase angles are 0, $\pi/2$, $-\pi/2$
- Reactance" means ratio of peak voltage to peak current (generalized resistances).







V_R & i_R in phase

Resistance

$$V_{max} / i_R \equiv R$$

V_C lags i_C by $\pi/2$

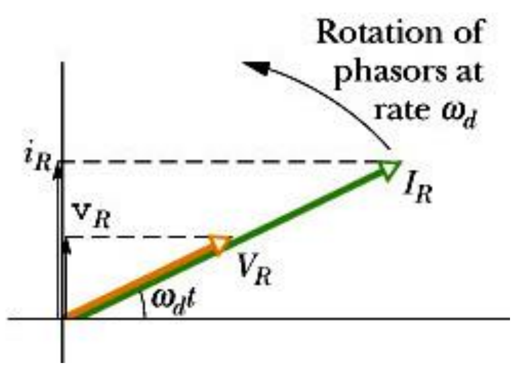
Capacitive Reactance

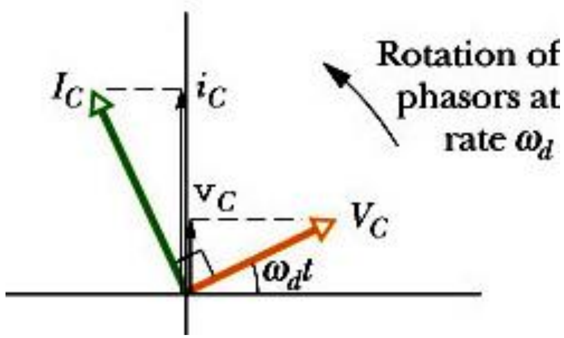
$$V_{max} / i_C \equiv X_C = \frac{1}{\omega_D C}$$

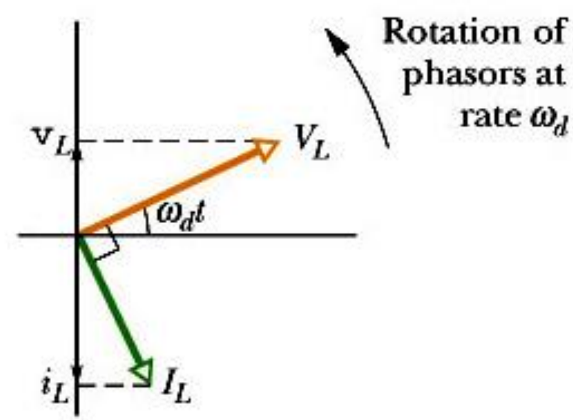
V_L leads i_L by $\pi/2$

Inductive Reactance

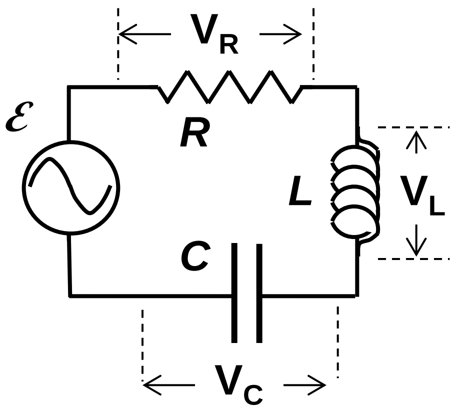
$$V_{max} / i_L \equiv X_L = \omega_D L$$







Series LCR circuit driven by an external AC voltage



Apply EMF:

$\mathcal{E}(t) = \mathcal{E}_{max}\sin(\omega_D t)$ ω_D is the driving frequency

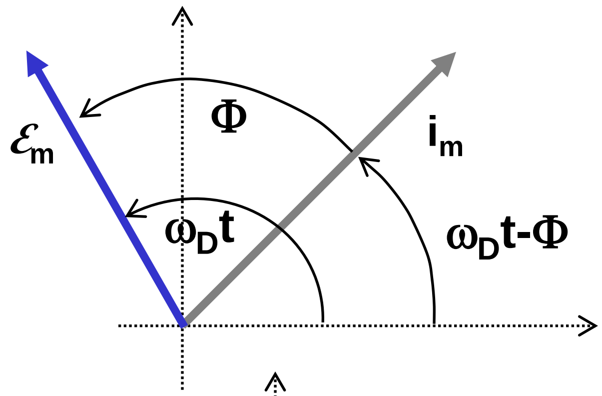
The current i(t) is the same everywhere in the circuit

$i(t) = i_{max}\sin(\omega_D t - \Phi)$

- Same frequency dependance as $\mathcal{E}(t)$ but...
- Current leads or lags $\mathcal{E}(t)$ by a constant phase angle Φ
- Same phase for the current in $\mathcal{E}$, R, L, & C

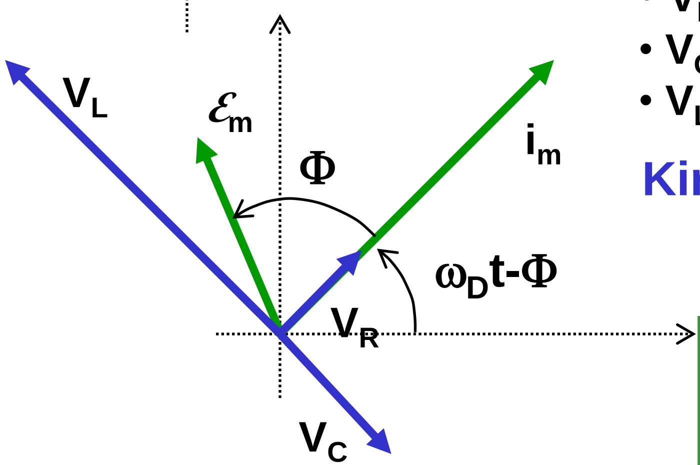
Phasors all rotate CCW at frequency ω_D

- Lengths of phasors are the peak values (amplitudes)
- The "y" components are the measured values.



Plot voltages in components with phases relative to current phasor i_m :

- V_R has same phase as i_m $\quad V_R = i_m R$
- V_C lags i_m by $\pi/2$ $\quad V_C = i_m X_C$
- V_L leads i_m by $\pi/2$ $\quad V_L = i_m X_L$



Kirchoff Loop rule for potentials (measured along y)

$$\vec{\mathcal{E}}(t) - \vec{V}_R(t) - \vec{V}_L(t) - \vec{V}_C(t) = 0$$

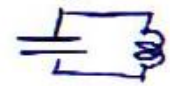
$$\vec{\mathcal{E}}_m = \vec{V}_R + (\vec{V}_L + \vec{V}_C) \qquad \vec{V}_C \text{ lags } \vec{V}_L \text{ by } 180^0$$

along i_m ↗ ↖ *perpendicular to i_m*

LC OSCILLATOR

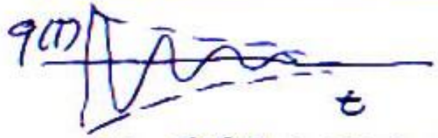
$q(t) = q_0 \cos(\omega t)$ $\quad \omega = (LC)^{-\frac{1}{2}}$ $\quad U_{TOT} = U_B + U_E = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}$

$I_m = q_0\omega$ $\quad i(t) = -q_0\omega\sin(\omega t)$ $\quad \frac{dU_{TOT}}{dt} = 0$



DAMPED LC OSCILLATOR

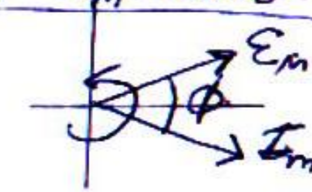
$q(t) = e^{-Rt/2L}\cos(\omega' t + \phi)$ $\quad \omega' = [\omega^2 - (\frac{R}{2L})^2]^{\frac{1}{2}}$ $\quad \omega = \sqrt{\frac{1}{LC}}$



UNDERDAMPED

AC CIRCUITS: APPLY EMF: $\mathcal{E}(t) = \mathcal{E}_0 \sin(\omega_d t)$ CURRENT: $i(t) = I_m \sin(\omega_d t - \phi)$

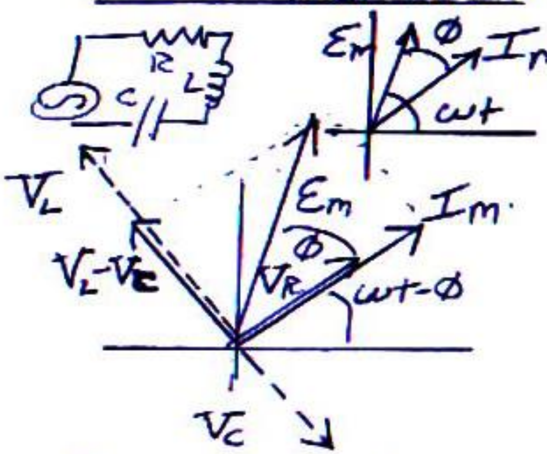
ϕ ≡ PHASE ANGLE RELATES PEAKS



PHASE & AMPLITUDE RELATIONS

RESISTOR R	Resistance R	I_m, V_R INPHASE	$V_R = I_R R$
CAPACITOR C	$X_C = (\omega C)^{-1}$	V_C LAGS I_m by $\pi/2$	$V_C = I_C X_C$
INDUCTOR L	$X_L = \omega L$	V_L LEADS I_m b $\pi/2$	$V_L = I_L X_L$.

SERIES LCR CIRCUIT: APPLY LOOP RULE, ALSO CURRENT THE SAME EVERYWHERE

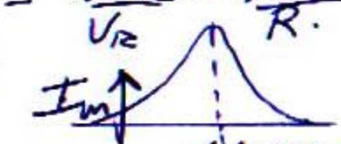


$\vec{\mathcal{E}}_m = \vec{V}_L - \vec{V}_C + \vec{V}_R$

$|\mathcal{E}_m| = [V_R^2 + (V_L - V_C)^2]^{\frac{1}{2}} = |I_m| Z$ ← IMPEDANCE.

$Z \equiv [R^2 + (X_L - X_C)^2]^{\frac{1}{2}}$ $\quad \tan\phi = \frac{V_L - V_C}{V_R} = \frac{X_L - X_C}{R}$.

RESONANCE AT $\omega_0 = \omega = 1/\sqrt{LC}$



AVERAGE POWER

$P_{AV} = I_{RMS}^2 R$

$I_{RMS} \equiv I_m/\sqrt{2}$

$\mathcal{E}_{RMS} \equiv \mathcal{E}_m/\sqrt{2}$, etc.

POWER FACTOR

$P_{AV} = \vec{\mathcal{E}}_{RMS} \cdot \vec{I}_{RMS}$

$= \mathcal{E}_{RMS} I_{RMS} \cos\phi$.

$\cos\phi = R/Z$

$R = Z\cos\phi$

$I_{RMS} = \frac{\mathcal{E}_{RMS}}{Z}$.

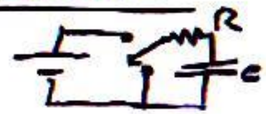
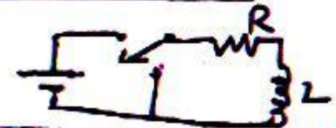
TRANSFORMERS

SAME FLUX IN EACH TURN

$\frac{N_S}{N_P} = \frac{V_S}{V_P} = \frac{I_P}{I_S}$

IDEAL XFORMER.

Summary: RC and RL circuit results

RECAP	RC CIRCUITS	LR CIRCUITS
TIME CONSTANT	$\tau_C = RC$ 	$\tau_L = L/R$ 
MEASURE OF CAPACITY	$C \equiv Q/V_C$ (GEOMETRY)	$L \equiv \frac{N\Phi_B}{i}$ = "FLUX LINKAGE" / UNIT CURRENT; Φ_B = FLUX THROUGH ONE COIL
LIMITING VALUES	$Q_\infty = C\mathcal{E}$	$i_\infty = \mathcal{E}/R$.
LOOP RULE VOLTAGE DROP	$V_C = Q/C$.	$\mathcal{E}_L = -L\,di/dt$ (FARADAY'S LAW)
EQUATION	$\mathcal{E} - R\frac{dQ}{dt} - \frac{Q}{C} = 0$	$\mathcal{E} - iR - L\frac{di}{dt} = 0$
GROWTH PHASE SOLUTION	$i(t) = \frac{\mathcal{E}}{R} e^{-t/\tau_C}$ $V_C(t) = -\mathcal{E}(1 - e^{-t/\tau})$ $V_R(t) = i(t)\,R$. $Q(t) = Q_\infty(1 - e^{-t/\tau})$	$i(t) = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau})$ $V_L(t) = -\mathcal{E}\,e^{-t/\tau}$ (BACK EMF) $V_R(t) = i(t)R$
DECAY PHASE SOLUTION	$i(t) = \frac{\mathcal{E}}{R} e^{-t/\tau}$ $V_C(t) = \mathcal{E}\,e^{-t/\tau}$ $Q(t) = C\mathcal{E}\,e^{-t/\tau}$	$i(t) = \frac{\mathcal{E}}{R} e^{-t/\tau}$ $V_L(t) = i(t)R = \mathcal{E}\,e^{-t/\tau}$.