Physics 121 - Electricity and Magnetism
Lecture 3 - Electric Field
Y&F Chapter 21 Sec. 4 – 7

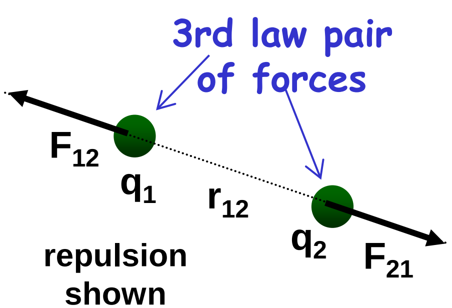
- **Recap & Definition of Electric Field**
- **Electric Field Lines**
- **Charges in External Electric Fields**
- **Field due to a Point Charge**
- **Field Lines for Superpositions of Charges**
- **Field of an Electric Dipole**
- **Electric Dipole in an External Field: Torque and Potential Energy**
- **Method for Finding Field due to Charge Distributions**
  - Infinite Line of Charge
  - Arc of Charge
  - Ring of Charge
  - Disc of Charge and Infinite Sheet
- **Motion of a charged paricle in an Electric Field - CRT example**

Recap: Electric charge

Basics:

- Positive and negative flavors. Like charges repel, opposites attract
- Charge is conserved and quantized. $e = 1.6 \times 10^{-19}$ Coulombs
- Ordinary matter seeks electrical neutrality – screening
- In conductors, charges are free to move around
  - screening and induction
- In insulators, charges are not free to move around
  - but materials polarize

Coulombs Law: forces at a distance enabled by a field



Constant
$k = 8.99 \times 10^9\ \mathrm{Nm^2/coul^2}$

Force on q_1 due to q_2 (magnitude)

$$\vec{F}_{12} = \frac{k\ q_1 q_2}{r_{12}^2}\hat{r}$$

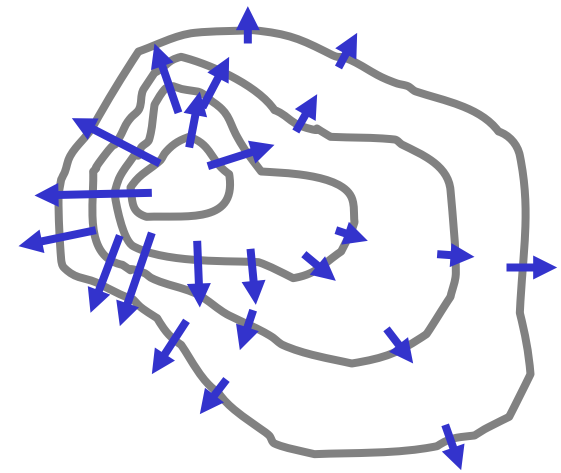
Superposition of Forces or Fields

$$\vec{F}_{\text{net on }1} = \sum_{i=2}^{n} \vec{F}_{1,i} = \vec{F}_{1,2} + \vec{F}_{1,3} + \vec{F}_{1,4} + \dots$$

Fields

Scalar Field Examples:

- Temperature - T(r)
- Pressure - P(r)
- Gravitational Potential energy – U(r)
- Electrostatic potential – V(r)
- Electrostatic potential energy – U(r)



Vector Field Examples:

- Velocity - v(r)
- Gravitational field/acceleration - g(r)
- Electric field – E(r)
- Magnetic field– B(r)
- Gradients of scalar fields

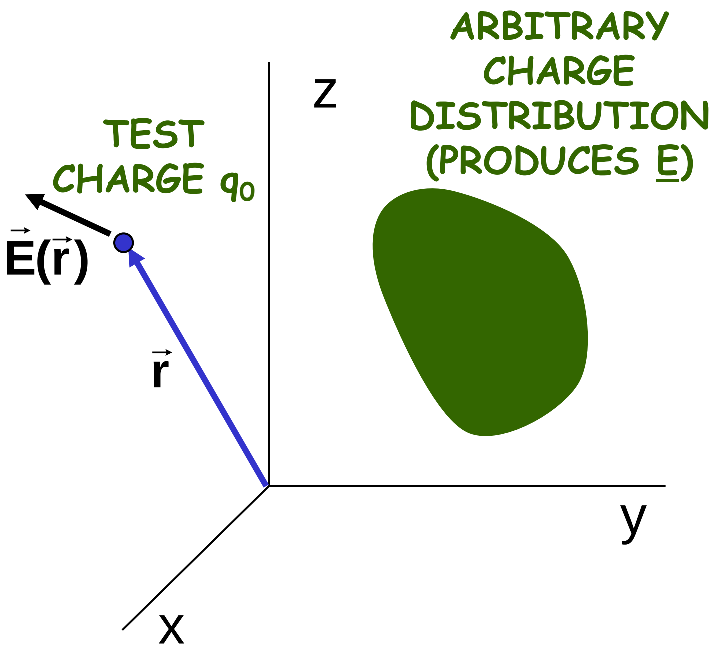
Fields "explain" forces at a distance – space altered by source

Gravitational Field versus **Electrostatic Field**

Gravitational Field	Electrostatic Field
force/unit mass	force/unit charge
$\vec{g}(\vec{r}) = \lim_{m_0 \to 0} \left(\frac{\vec{F}_g(\vec{r})}{m_0} \right)$	$\vec{E}(\vec{r}) = \lim_{q_0 \to 0} \left(\frac{\vec{F}_e(\vec{r})}{q_0} \right)$
m_0 is a "test mass"	q_0 is a positive "test charge"

"Test" masses or charges map the direction and magnitudes of fields

Field due to a charge distribution



Test charge q_0:

- small and positive
- does not affect the charge distribution that produces E.

A charge distribution creates a field:

- Map E field by moving q_0 around and measuring the force F at each point
- E(r) is a vector parallel to F(r)
- E field exists whether or not the test charge is present
- E varies in direction and magnitude

$$\vec{E}(\vec{r}) \equiv \lim_{q_0 \to 0} \left(\frac{\vec{F}(\vec{r})}{q_0} \right)$$

most often...

$$\vec{E}(\vec{r}) = \frac{\vec{F}(\vec{r})}{q_0}$$

$$\vec{F}(\vec{r}) = q_0 \vec{E}(\vec{r})$$

F = Force on test charge q_0 at point r due to the charge distribution

E = External electric field at point r = Force / unit charge

SI Units: Newtons / Coulomb later: V / m

Electrostatic Field Examples

Field Location	Value
Inside copper wires in household circuits	10^{-2} N/C
Near a charged comb	10^{3} N/C
Inside a TV picture tube (CRT)	10^{5} N/C
Near the charged drum of a photocopier	10^{5} N/C
Breakdown voltage across an air gap (arcing)	3×10^{6} N/C
E-field at the electron's orbit in a hydrogen atom	5×10^{11} N/C
E-field on the surface of a Uranium nucleus	3×10^{21} N/C

$$\vec{E} = \frac{\vec{F}}{q_0}$$

- **Magnitude:** $E=F/q_0$
- **Direction:** same as the force that acts on the positive test charge
- **SI unit:** N/C

Electric Field due to a point charge Q

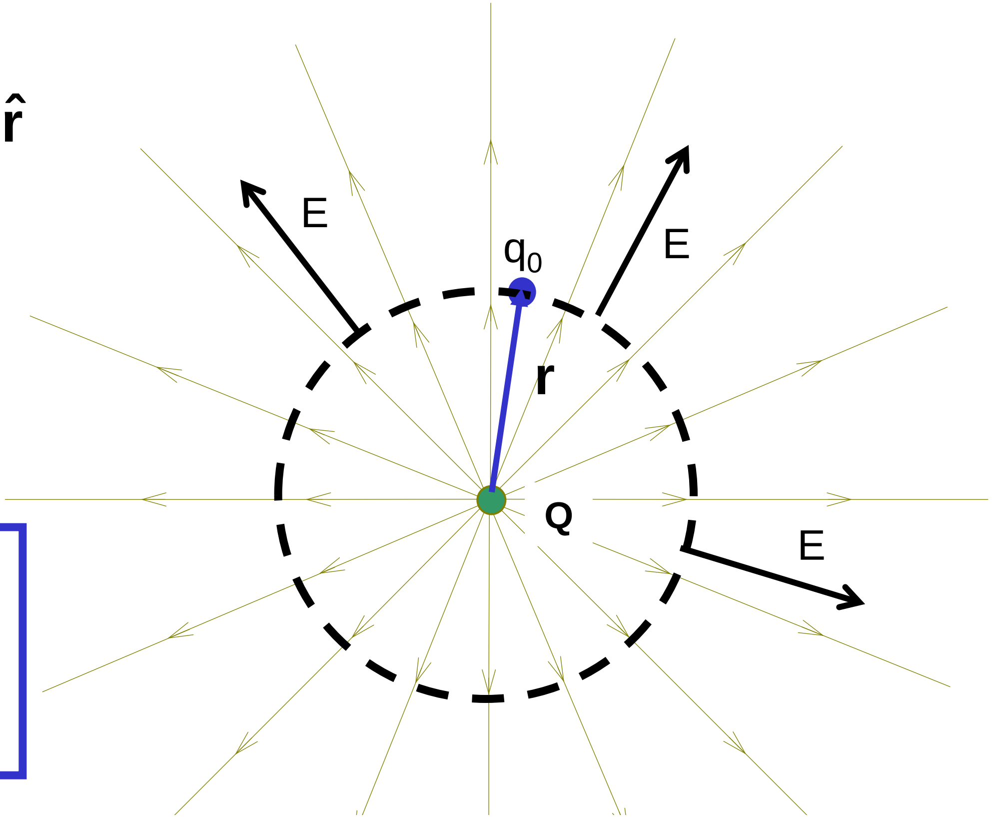
Coulombs Law test charge q_0

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq_0}{r^2}\hat{r}$$

Find the field $\underline{E}$ due to point charge Q as a function over all of space

$$\vec{F} = q_0\vec{E} \Rightarrow$$

$$\vec{E} \equiv \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r} \qquad \hat{r} \equiv \frac{\vec{r}}{r}$$

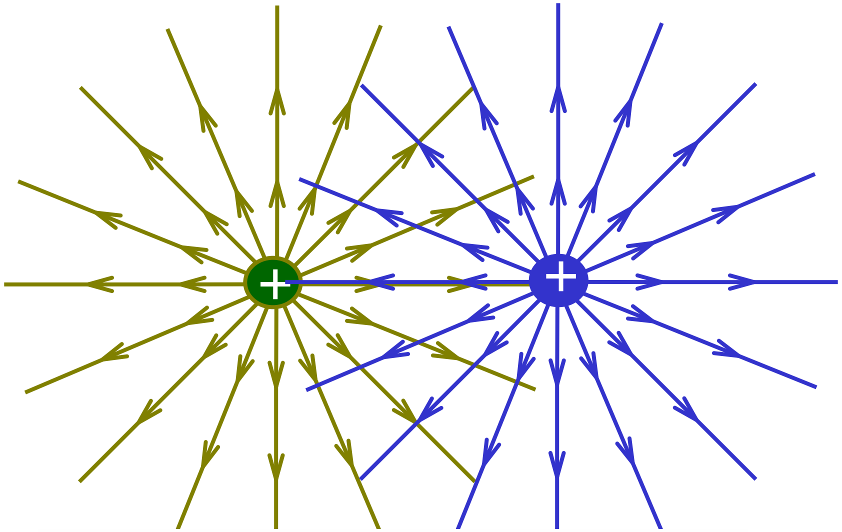


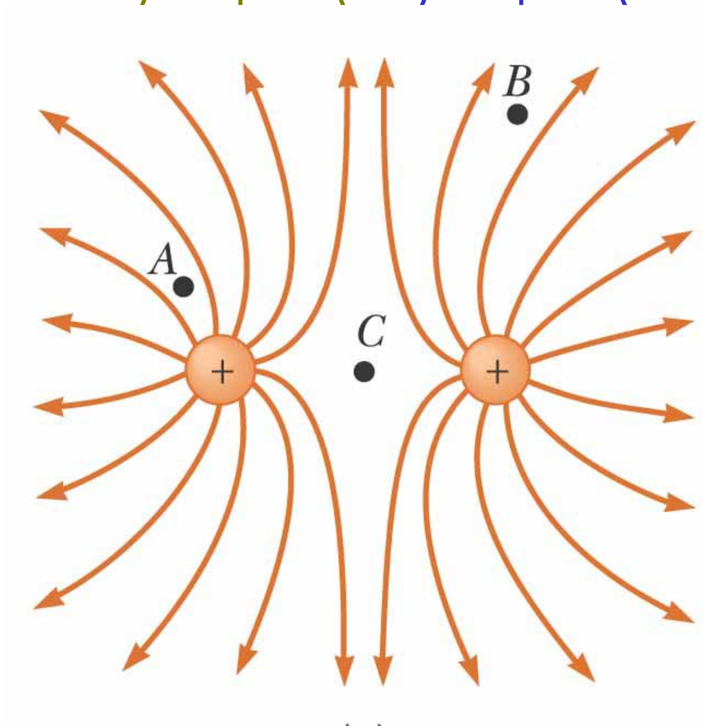
- Magnitude $E = KQ/r^2$ is constant on any spherical shell (spherical symmetry)
- Visualize: E field lines are radially out for +|Q|, in for -|Q|
- Flux through any closed (spherical) shell enclosing Q is the same:

$$\Phi = EA = Q.4\pi r^2/4\pi\varepsilon_0 r^2 = Q/\varepsilon_0$$ Radius cancels

The closed (Gaussian) surface intercepts all the field lines leaving Q

Use superposition to calculate net electric field at each point due to a group of individual charges





Example: for point charges at $\underline{r}_1$, $\underline{r}_2$.....

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n$$

$$\vec{E}_{net} = \frac{\vec{F}_{tot}}{q_0} = \frac{\vec{F}_1}{q_0} + \frac{\vec{F}_2}{q_0} + \ldots + \frac{\vec{F}_n}{q_0}$$

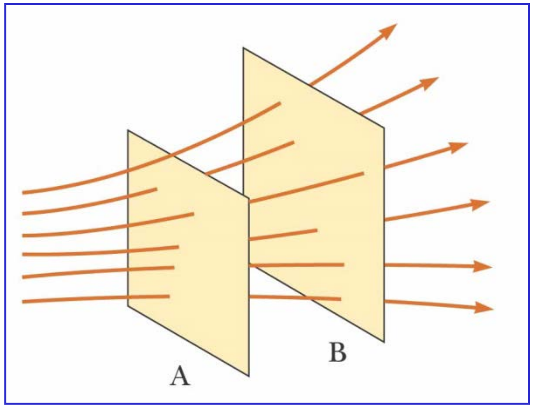
$$= \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_n$$

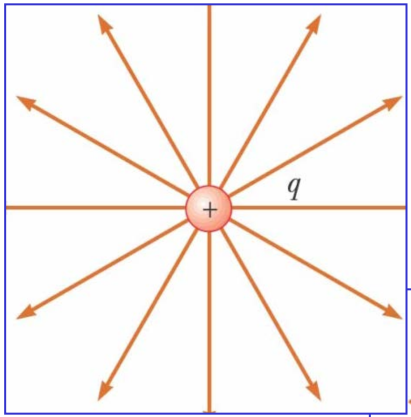
$$\boxed{\vec{E}_{\text{net at i}} = \frac{1}{4\pi\varepsilon_0} \sum_j \frac{q_i}{r_{ij}^{\,2}} \hat{r}_{ij}}$$

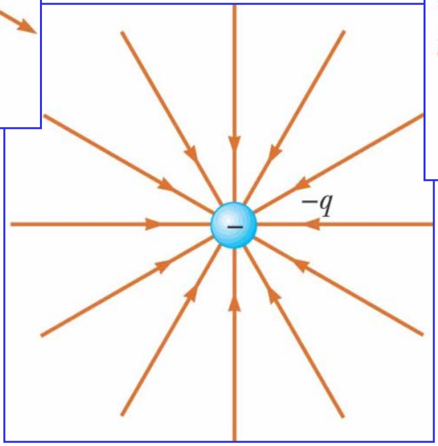
Do the sum above for every test point i

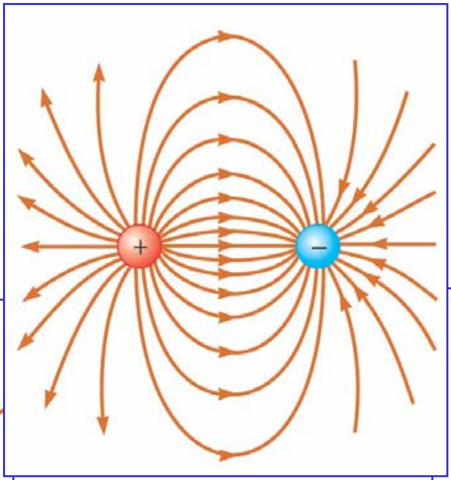
Visualization: Electric field lines (Lines of force)

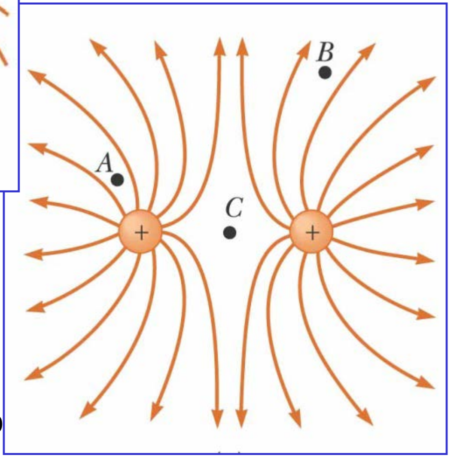
- Map direction of an electric field line by moving a positive test charge around.
- The tangent to a field line at a point shows the field direction there.
- The density of lines crossing a unit area perpendicular to the lines measures the strength of the field. Where lines are dense the field is strong.
- Lines begin on positive charges (or infinity and end on negative charges (or infinity).
- Field lines cannot cross other field lines



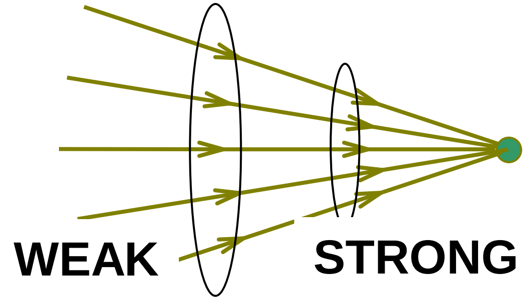






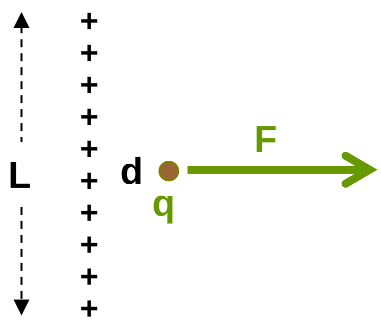


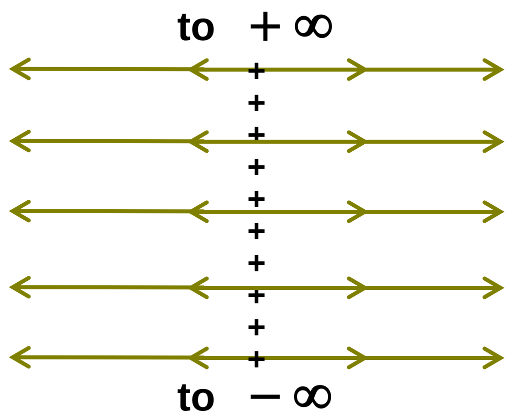
DETAIL NEAR A POINT CHARGE



NEAR A LARGE, UNIFORM SHEET OF + CHARGE

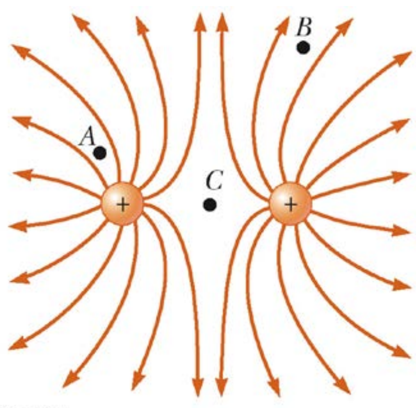
- No conductor – just an infinitely large charge sheet
- E approximately constant in the "near field" region (d << L)



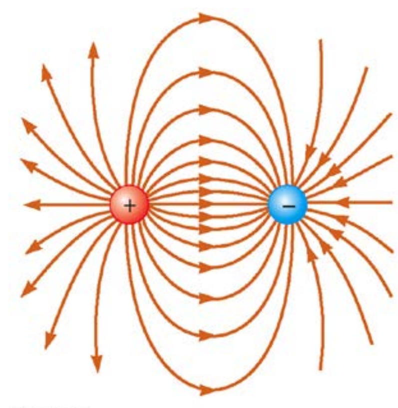


The field has uniform intensity & direction everywhere except on sheet

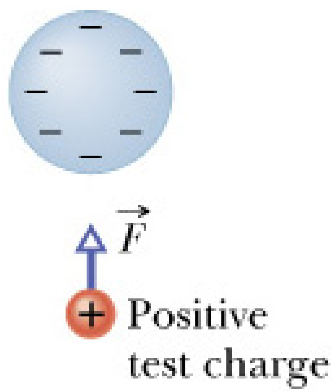
TWO EQUAL + CHARGES (REPEL)

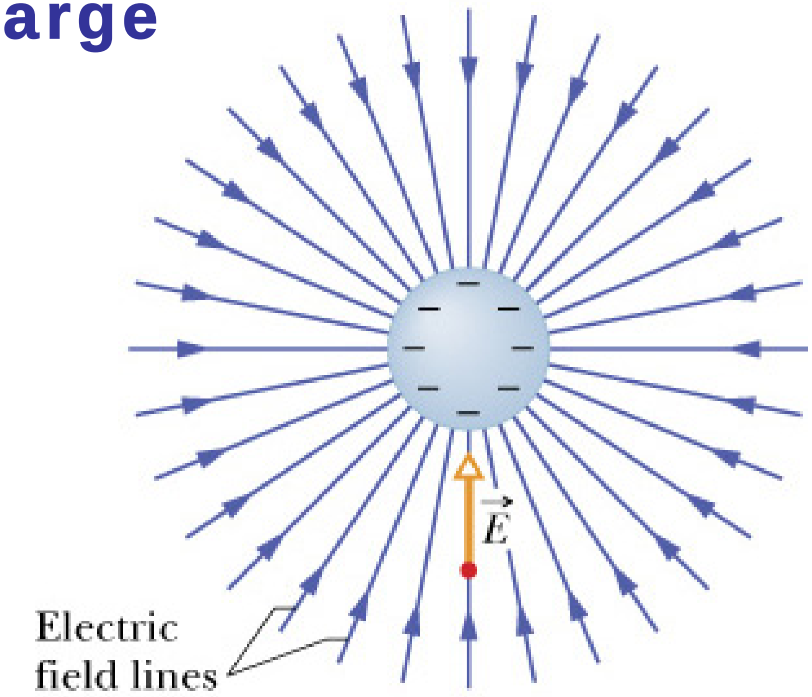


EQUAL + AND – CHARGES ATTRACT



Field lines for a spherical shell or solid sphere of charge





(b)

Shell Theorem Conclusions

Outside point:

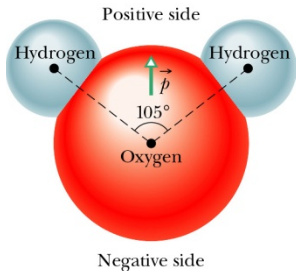
Same field as point charge

Inside spherical distribution at distance r from center:

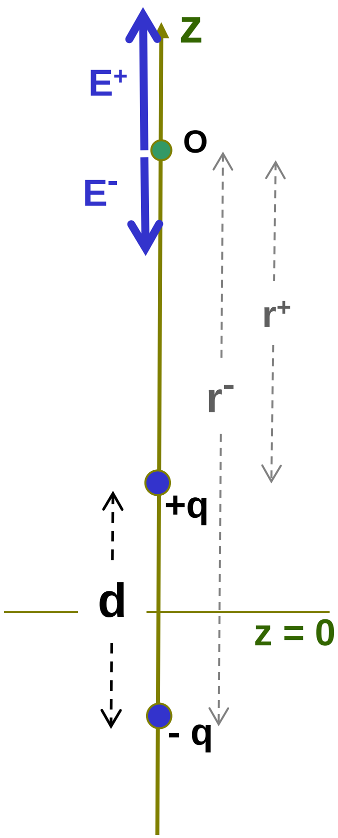
- **E = 0 for hollow shell;**
- **$E = kQ_{inside}/r^2$ for solid sphere**

Example: Find E_{net} at a point on the axis of a dipole

- Use superposition
- Symmetry → E_{net} parallel to z-axis



$$r^+ \equiv z - d/2 \quad \text{and} \quad r^- \equiv z + d/2$$



$$E_{\text{at O}} = E^+ - E^- = \frac{kq}{r_+^2} - \frac{kq}{r_-^2}$$

- Limitation: z > d/2 or z < - d/2

$$E_{\text{at O}} = kq\left[\frac{1}{(z-d/2)^2} - \frac{1}{(z+d/2)^2}\right]$$

DIPOLE MOMENT

$$\vec{p} \equiv q\vec{d}$$

points from – to +

$+q$

$\vec{d}$

$-q$

$$E_{\text{at O}} = 2kqd\left[\frac{z}{(z^2 - d^2/4)^2}\right]$$

Exact

For z >> d : point "O" is "far" from center of dipole

$$[\ ] \approx \frac{1}{z^3} \quad \text{since} \quad \frac{d}{z} << 1$$

$$\therefore E_{\text{at O}} \approx +\frac{qd}{2\pi\varepsilon_0}\frac{1}{z^3} = +\frac{p}{2\pi\varepsilon_0}\frac{1}{z^3}$$

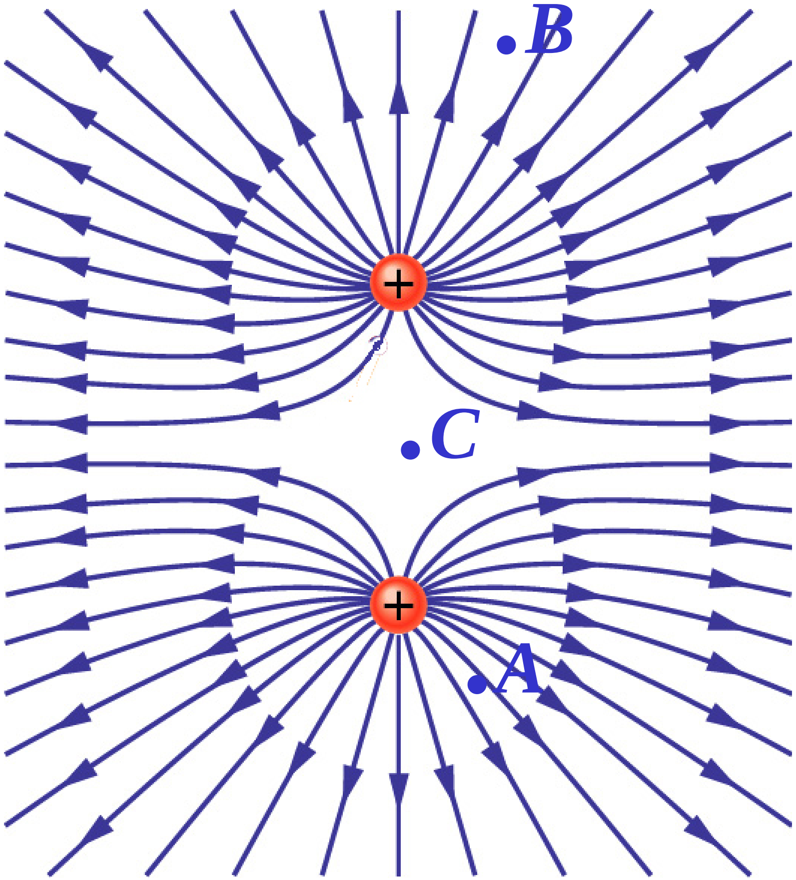
Exercise: Do these formulas describe E at the point midway between the charges
Ans: $E = -4p/2\pi\varepsilon_0 d^3$

- Fields cancel as d → 0 so E→ 0
- E falls off as $1/z^3$ not $1/z^2$
- E is negative when z is negative
- Does "far field" E look like point charge?

Electric Field

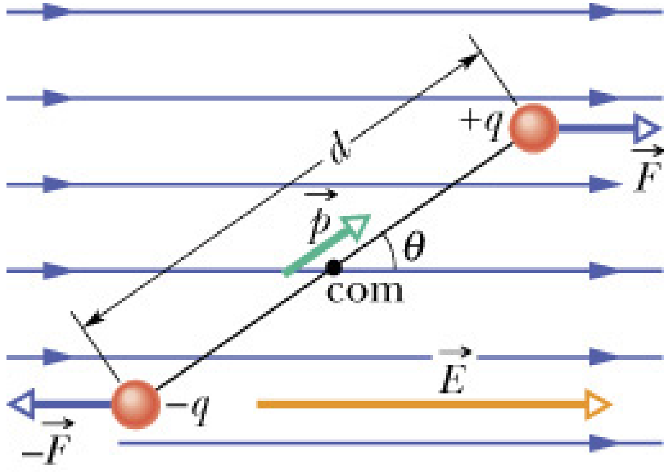
3-2: Put the magnitudes of the electric field values at points A, B, and C shown in the figure in decreasing order.

A) $E_C > E_B > E_A$
B) $E_B > E_C > E_A$
C) $E_A > E_C > E_B$
D) $E_B > E_A > E_C$
E) $E_A > E_B > E_C$



A Dipole in a Uniform EXTERNAL Electric Field

Feels torque - Stores potential energy (See Sec 21.7)



ASSSUME RIGID DIPOLE

$$\vec{p} \equiv q\vec{d}$$

Dipole Moment Vector

$$\vec{\tau} = \vec{p} x \vec{E}$$

Torque = Force x moment arm
= - 2 q E x (d / 2) sin(θ)
= - p E sin(θ)
(CW, into paper as shown)

- |torque| = 0 at θ = 0 or θ = π
- |torque| = pE at θ = +/- π/2
- RESTORING TORQUE: $\tau(-\theta) = \tau(+\theta)$

Potential Energy U = -W

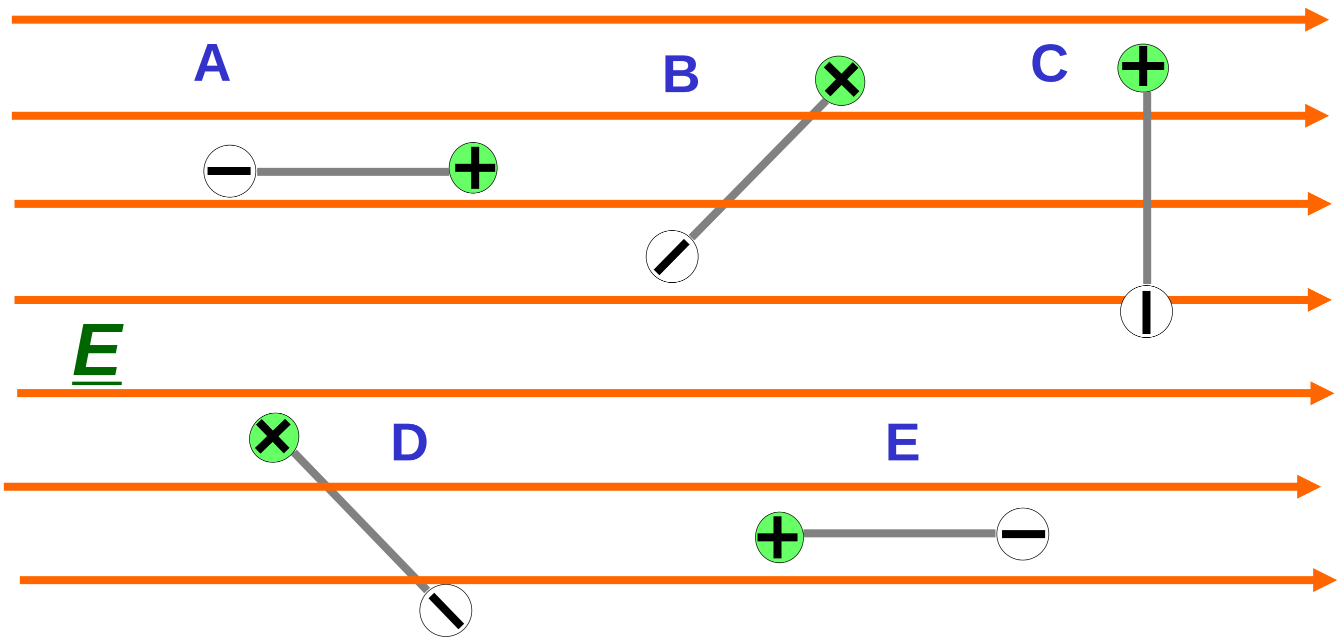
$$U = -\int \tau d\theta = +pE\int \sin(\theta) d\theta$$
$$= -pE\cos(\theta)$$

$$U_E = -\vec{p} \cdot \vec{E}$$

OSCILLATOR

- U = 0 for θ = +/- π/2
- U = - pE for θ = 0 minimum
- U = + pE for θ = π maximum

3-3: In the sketch, a dipole is free to rotate in a uniform external electric field. Which configuration has the smallest potential energy?



3-4: Which configuration has the largest potential energy?

Method for finding the electric field at point P - - given a known *continuous* charge distribution

This process is just superposition

$$\vec{E}_P = \frac{1}{4\pi\varepsilon_0} \lim_{\Delta q \to 0} \sum_i \frac{\Delta q_i}{r_i^2} \hat{r}_i \Rightarrow \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r^2} \hat{r}$$

1. Find an expression for dq, the "point charge" within a differentially "small" chunk of the distribution

$$dq = \begin{cases} \lambda dl & \text{for a linear distribution} \\ \sigma dA & \text{for a surface distribution} \\ \rho dV & \text{for a volume distribution} \end{cases}$$

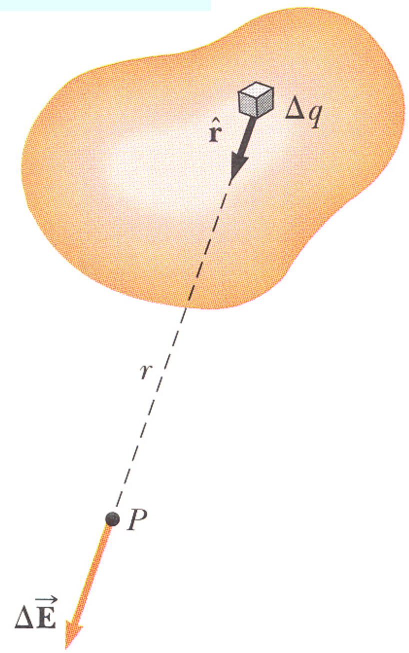
2. Represent field contributions at P due to a point charge dq located anyhwere in the distribution. Use symmetry where possible.

$$\Delta\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\Delta q}{r^2} \hat{r} \quad \Rightarrow \quad d\vec{E} = \frac{dq}{4\pi\varepsilon_0 r^2} \hat{r}$$

3. Add up (integrate) the contributions d$\underline{E}$ over the whole distribution, varying the displacement and direction as needed.

Use symmetry where possible.

$$\vec{E}_P = \int_{dist} d\vec{E} \qquad \text{(line, surface, or volume integral)}$$

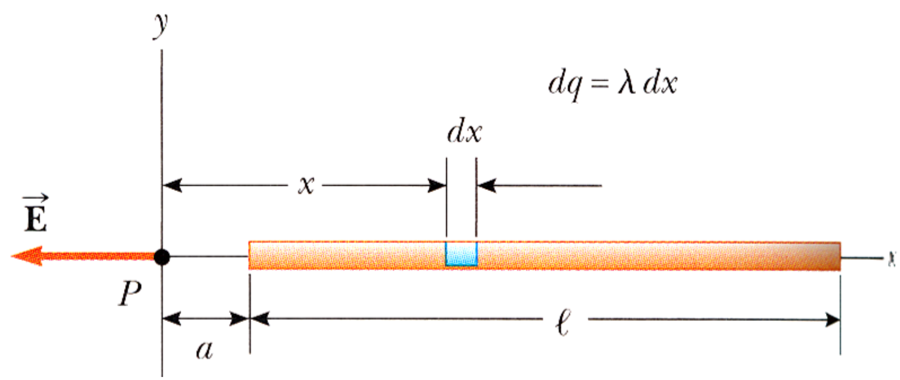


Example: Find electric field on the axis of a charged rod

- **Rod has length L, uniform positive charge per unit length λ, total charge Q. $\lambda = Q/L$.**
- **Calculate electric field at point P on the axis of the rod a distance a from one end. Field points along x-axis.**

$$dq = \lambda dx$$

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{x^2}$$



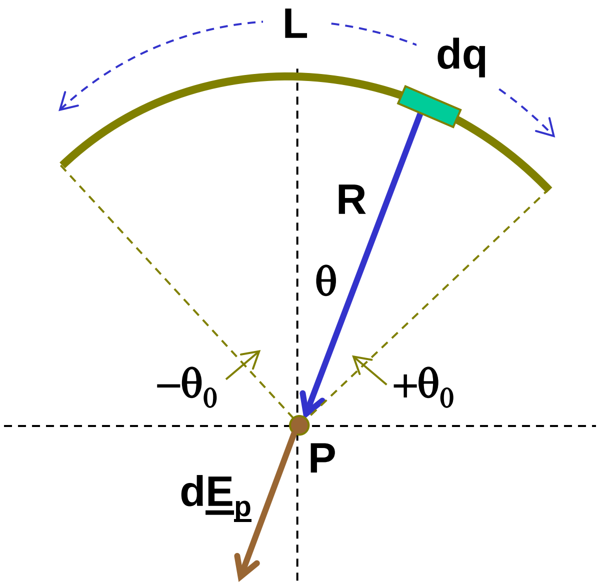
- **Add up contributions to the field from all locations of dq along the rod ($x \in [a, L + a]$).**

$$E = \int_a^{L+a} \frac{\lambda}{4\pi\varepsilon_0}\frac{dx}{x^2} = \frac{\lambda}{4\pi\varepsilon_0}\int_a^{L+a}\frac{dx}{x^2} = \frac{\lambda}{4\pi\varepsilon_0}\left[-\frac{1}{x}\right]_a^{L+a} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{L}\left(\frac{1}{a} - \frac{1}{L+a}\right)$$

$$\therefore \quad E = \frac{Q}{4\pi\varepsilon_0 a(L+a)}$$

- Interpret Limiting cases:
  - L => 0 rod becomes point charge
  - L << a same, L/a << 1
  - L >> a a/L << 1,

Electric field at center of an ARC of charge



- Uniform linear charge density $\lambda = Q/L$

$$dq = \lambda ds = \lambda R d\theta$$

- P on symmetry axis at center of arc
  → Net E is along y axis → need E_y only

$$d\vec{E}_P = \frac{k dq}{R^2}\hat{r} \quad \rightarrow \quad dE_{P,y} = -\frac{k\, dq \cos(\theta)}{R^2}\hat{j}$$

- Angle θ is between $-\theta_0$ and $+\theta_0$

- Integrate:

$$\vec{E}_{P,y} = \frac{-1}{4\pi\varepsilon_0}\hat{j}\frac{\lambda}{R^2}R\int_{-\theta_0}^{+\theta_0}\cos(\theta)d\theta = -\frac{\lambda}{4\pi\varepsilon_0}\hat{j}\frac{1}{R}\sin(\theta)\Big|_{-\theta_0}^{+\theta_0}$$

note: $\int \cos(\theta)d\theta = \sin(\theta)$ and $\sin(-\theta) = -\sin(\theta)$

$$\therefore\ \vec{E}_{P,y} = -2k\frac{\lambda}{R}\sin(\theta_0)\,\hat{j}$$

In the plane of the arc

- For a semi-circle, $\theta_0 = \pi/2$

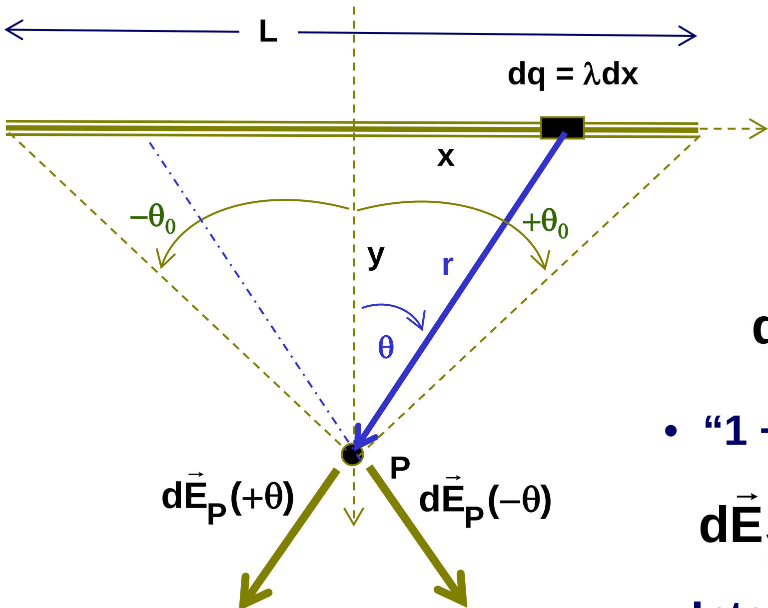
$$\vec{E}_{P,y} = -2k\frac{\lambda}{R}\ \hat{j}$$

- For a full circle, $\theta_0 = \pi$

$$\vec{E}_{P,y} = 0$$

Electric field due to a straight LINE of charge
Point P on symmetry axis, a distance y off the line

SEE Y&F Example 21.10



- uniform linear charge density: $\lambda = Q/L$
- point "P" is at y on symmetry axis
- by symmetry, total E is along y-axis
  x-components of dE pairs cancel
- solve for line segment, then let y << L

$$d\vec{E}_P = \frac{k\,dq}{r^2}\hat{r} \quad \rightarrow \quad dE_{P,y} = -\frac{k\,dq\cos(\theta)}{r^2}\hat{j}$$

$$x = y\tan(\theta)$$
$$r^2 = x^2 + y^2$$
$$\boxed{r^2 = y^2\,[1+\tan^2(\theta)]}$$

Find dx in terms of θ:

$$\frac{dx}{d\theta} = y\frac{d\,[\tan(\theta)]}{d\theta} = y\frac{d}{d\theta}\frac{\sin(\theta)}{\cos(\theta)}$$
$$= y\,[1+\tan^2(\theta)]$$
$$dx = y\,[1+\tan^2(\theta)]\,d\theta$$
$$\boxed{dq \equiv \lambda dx = \lambda y\,[1+\tan^2(\theta)]\,d\theta}$$

- "1 + tan²(θ)" cancels in numerator and denominator

$$d\vec{E}_{y,P} = -\frac{k\,\lambda\cos(\theta)d\theta}{y}\hat{j}$$

- Integrate from $-\theta_0$ to $+\theta_0$

$$\vec{E}_{y,P} = -\frac{k\,\lambda}{y}\hat{j}\int_{-\theta_0}^{+\theta_0}\cos(\theta)d\theta = -\frac{k\,\lambda}{y}\hat{j}\sin(\theta)\Big|_{-\theta_0}^{+\theta_0}$$

$$\boxed{\vec{E}_P = -\frac{2k\,\lambda}{y}\hat{j}\sin(\theta_0)}$$

Finite length wire

- For y << L (wire looks infinite) $\theta_0 \rightarrow \pi/2$

$$\boxed{\vec{E}_P = -\frac{2k\,\lambda}{y}\hat{j}}$$

Falls off as 1/y
Along −y direction

Cop

Electric field due to a RING of charge at point P on the symmetry (z) axis

SEE Y&F Example 21.9

- Uniform linear charge density along circumference: $\lambda = Q/2\pi R$
- $dq = \lambda ds$ = charge on arc segment of length $ds = R d\phi$
- P on symmetry axis → xy components of E cancel
- Net E field is along z only, normal to plane of ring

$$d\vec{E}_P = \frac{k dq}{r^2}\hat{r} \rightarrow dE_{z,P} = \frac{k\, dq \cos(\theta)}{r^2}\hat{k}$$

$$dq \equiv \lambda\, ds = \lambda\, R\, d\phi \qquad \cos(\theta) = z/r \qquad r^2 = R^2 + z^2$$

$$d\vec{E}_{P,z} = \frac{k\,\lambda R z d\phi}{r^3}\hat{k}$$

- Integrate on azimuthal angle ϕ from 0 to 2π

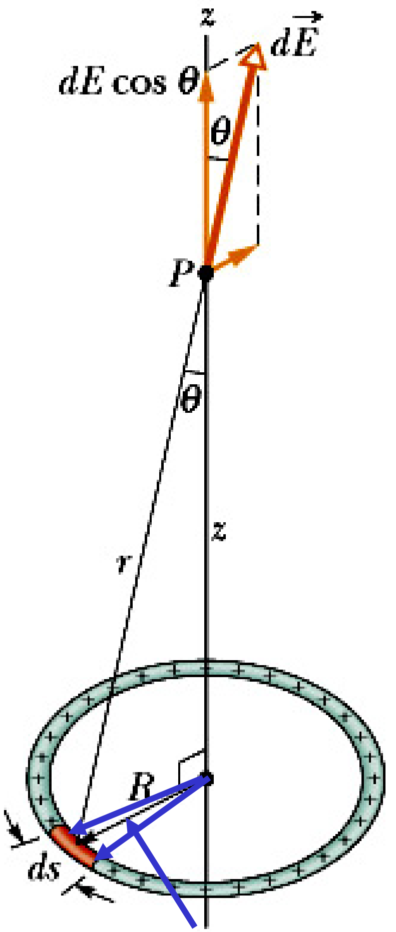
$$\vec{E}_{P,z} = \frac{k\,\lambda R z}{[R^2 + z^2]^{3/2}}\vec{k}\int_0^{2\pi} d\phi$$

integral = 2π

$$2\pi R\lambda \equiv Q$$ total charge on disk

$$\vec{E}_{P,z} = \frac{kQz}{[R^2 + z^2]^{3/2}}\hat{k}$$

E_z → 0 as z → 0 (see result for arc)



$ds = R\ d\phi$ $\quad d\phi$

- Limit: For P "far away" use z >> R

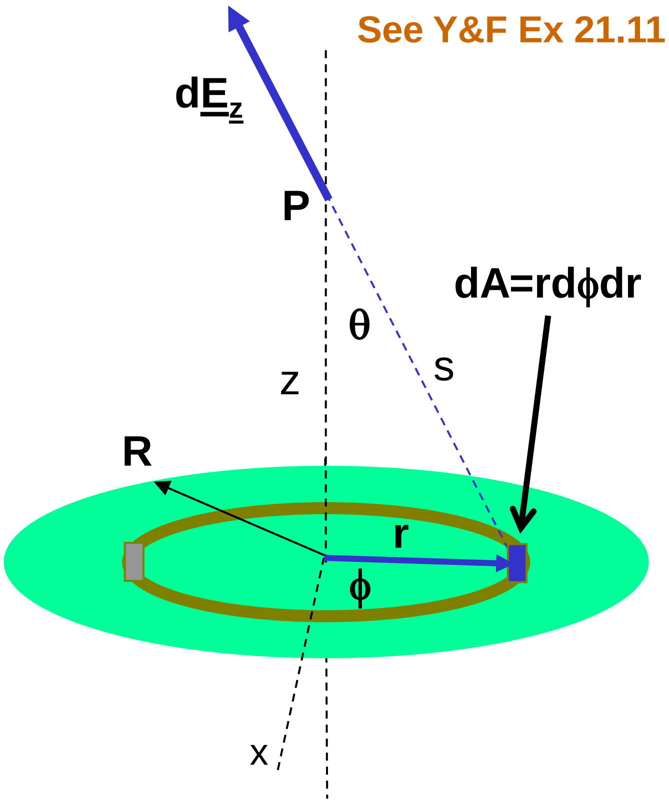
$$E_{P,z} \rightarrow \frac{kQ}{z^2}$$

Ring looks like a point charge if point P is very far away!

Exercise: Where is E_z a maximum?
Set $dE_z/dz = 0$
Ans: $z = R/\text{sqrt}(2)$

Electric field due to a DISK of charge for point P on z (symmetry) axis

See Y&F Ex 21.11



- Uniform surface charge density on disc in x-y plane
  $\sigma = Q/\pi R^2$
- Disc is a set of rings, each of them dr wide in radius
- P on symmetry axis → net E field only along z
- dq = charge on arc segment rdϕ with radial extent dr

$$dA = r\,dr\,d\phi \qquad dq \equiv \sigma\,dA = \sigma\,r\,dr\,d\phi$$

$$\cos(\theta) = z/s \qquad s^2 = r^2 + z^2$$

$$d\vec{E}_z = \frac{k\,dq}{s^2}\cos(\theta)\,\hat{k} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\,z\,r\,dr\,d\phi}{\left[r^2 + z^2\right]^{3/2}}\,\hat{k}$$

- Integrate twice: first on azimuthal angle ϕ from 0 to 2π which yields a factor of 2π then on ring radius r from 0 to R

$$\vec{E}_z = \frac{2\pi\sigma}{4\pi\varepsilon_0} z \int_0^R \frac{r\,dr}{[r^2 + z^2]^{3/2}}\,\vec{k}$$

Note Anti-derivative

$$\frac{r}{[r^2+z^2]^{3/2}} = \frac{d}{dr}\left\{\frac{-1}{[r^2+z^2]^{1/2}}\right\}$$

$$\vec{E}_{disk} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\left[z^2 + R^2\right]^{1/2}}\right]\hat{k}$$

Electric field due to a DISK of charge, continued

Exact Solution: $$\vec{E}_{disk} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\left[z^2 + R^2\right]^{1/2}}\right]\hat{k}$$

Near-Field: z<< R: P is close to the disk. Disk looks like infinite sheet.

$$\text{for } z/R \ll 1: \quad \vec{E}_{disk} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{R\left[1 + (z/R)^2\right]^{1/2}}\right]\hat{k} \approx \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{R}\right]\hat{k} \approx \frac{\sigma}{2\varepsilon_0}$$

$$\vec{E}_{disk} \approx \frac{\sigma}{2\varepsilon_0}\hat{k}$$

"near field" is constant – disk approximates an infinite sheet of charge

Far-Field: R<< z: P is far from to the disk. Disk looks like a point charge.

$$\text{for } R/z \ll 1: \quad \vec{E}_{disk} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{z\left[1 + (R/z)^2\right]^{1/2}}\right]\hat{k}$$

Recall $\sigma = Q/\pi R^2$

Series Expansion

$(1+s)^n = 1 + ns/1! + n(n-1)s^2/2! + \ldots$

converges quickly for $s \ll 1$

approximate: $$\frac{1}{\left[1 + (R/z)^2\right]^{1/2}} \approx 1 - \frac{1}{2}\left(\frac{R}{z}\right)^2 + \ldots$$

$$\therefore \quad \vec{E}_{disk} \approx \frac{\sigma}{2\varepsilon_0}\left[1 - 1 + \frac{1}{2}\frac{R^2}{z^2}\right]\hat{k} = \frac{\sigma}{4\varepsilon_0}\frac{R^2}{z^2}\hat{k}$$

$$\vec{E}_{disk} \approx \frac{Q}{4\pi\varepsilon_0}\frac{1}{z^2}\hat{k}$$

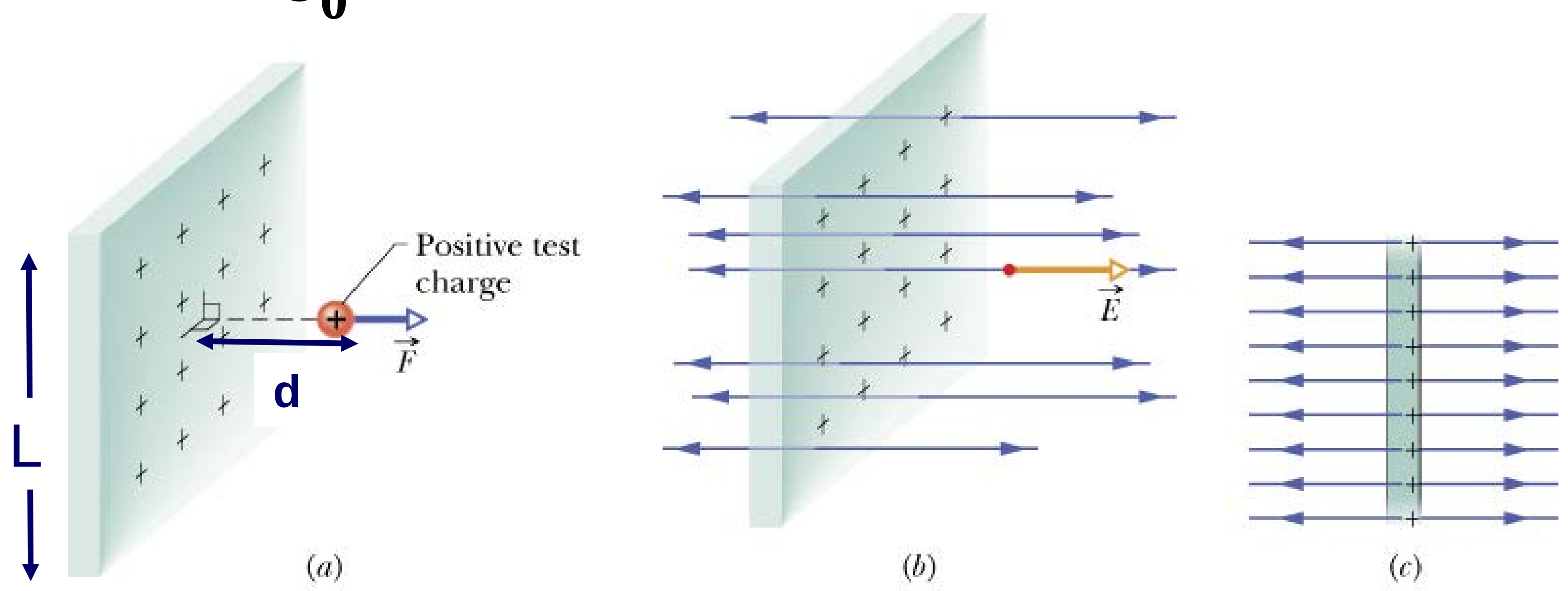
Point charge formula

Infinite (i.e."large") uniformly charged sheet

Non-conductor, fixed surface charge density σ

Infinite sheet → d<<L → "near field" → uniform field

$$E = \frac{\sigma}{2\varepsilon_0} \text{ for infinite, non-conducting charged sheet}$$



Method: solve non-conducting disc of charge for point on z-axis then approximate z << R

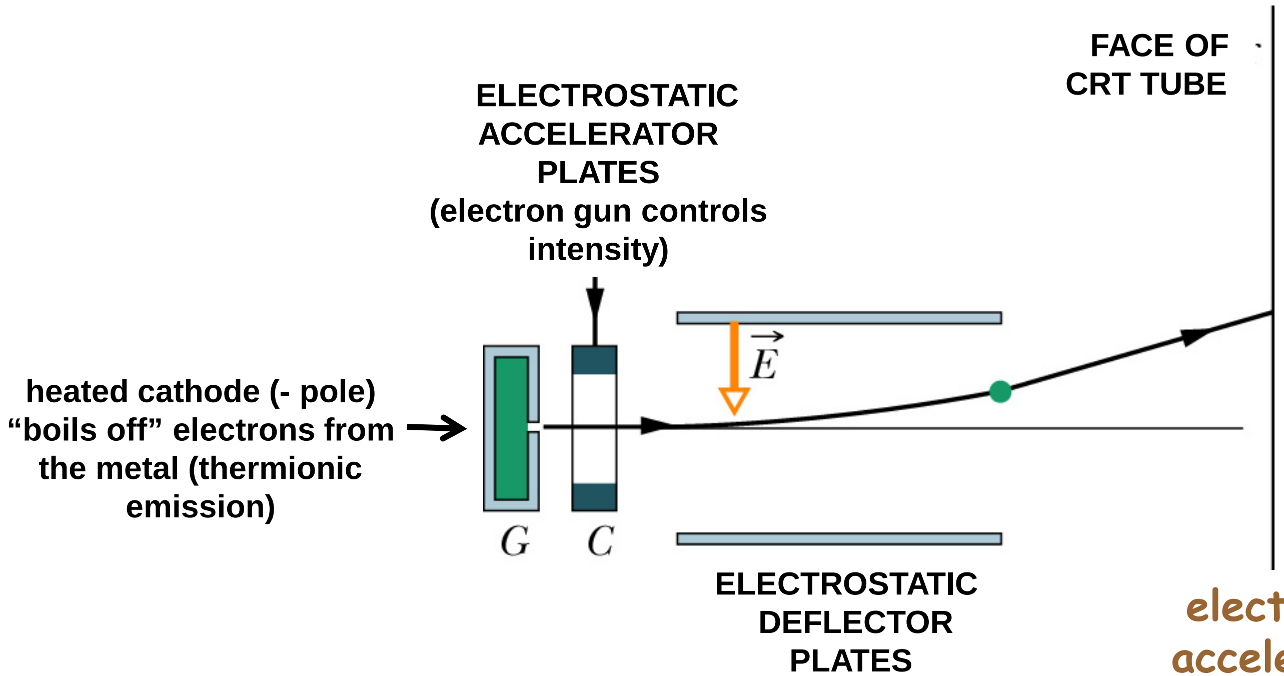
Motion of a Charged Particle in a Uniform Electric Field

$$\vec{F} = q\vec{E}$$

$$\vec{F}_{net} = m\vec{a}$$

- Stationary charges produce E field at location of charge q
- Acceleration a is parallel or anti-parallel to E.
- Acceleration is F / m not F / q = E
- Acceleration is the same everywhere in uniform field

Example: Early CRT tube with electron gun and electrostatic deflector

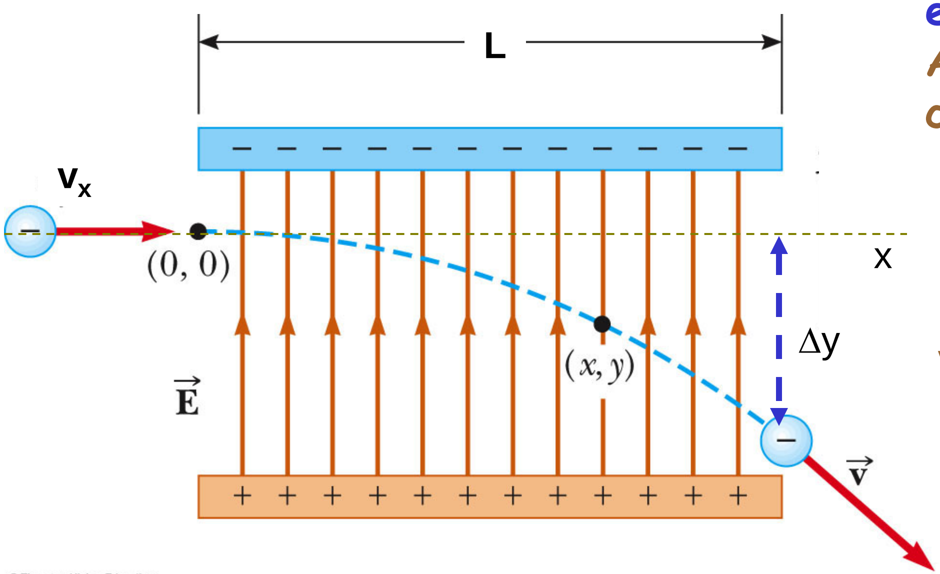


electrons are negative so acceleration a and electric force F are in the direction opposite the electric field E.

Motion of a Charged Particle in a Uniform Electric Field

$\vec{F} = q\vec{E}$ $\vec{F}_{net} = m\vec{a}$

Kinematics: ballistic trajectory



Δy is the DEFLECTION of the electron as it crosses the field

Acceleration has only a constant y component. v_x is constant, a_x=0

$$a_y = -\frac{eE}{m}$$

v_x yields time of flight Δt

$$v_x = \frac{L}{\Delta t}$$

electrons are negative so acceleration a and electric force F are in the direction opposite the electric field E.

Measure deflection, find Δt via kinematics. Evaluate v_y & v_x

$$\Delta y = \frac{1}{2} a_y \, \Delta t^2 = -\frac{1}{2}\frac{eE}{m}\Delta t^2$$

$$v_y = a_y \Delta t \qquad v = \left(v_x^2 + v_y^2\right)^{1/2}$$

Use to find e/m