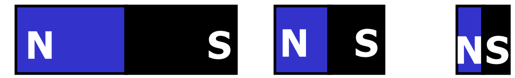
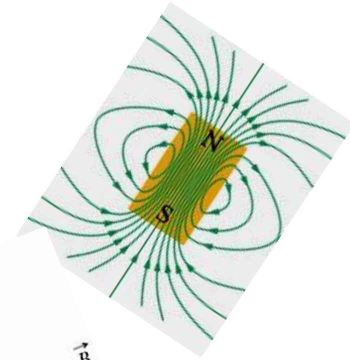
Electricity and Magnetism - Physics 121
Lecture 10 - Sources of Magnetic Fields (Currents)
Y&F Chapter 28, Sec. 1 - 7

- Magnetic fields are due to currents
- The Biot-Savart Law
- Calculating field at the centers of current loops
- Field due to a long straight wire
- Force between two parallel wires carrying currents
- Ampere’s Law
- Solenoids and toroids
- Field on the axis of a current loop (dipole)
- Magnetic dipole moment
- Summary

Previously: moving charges and currents feel a force in a magnetic field

- **Magnets come only as dipole pairs of N and S poles (no monopoles).**

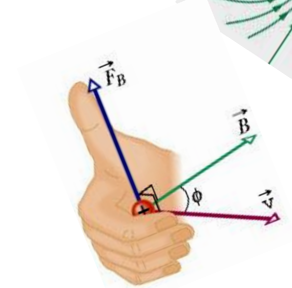




- **Magnetic field exerts a force on *moving* charges (i.e. on currents).**

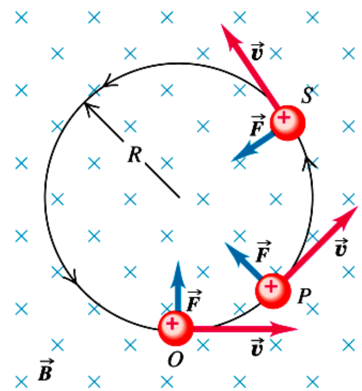
- **The force is perpendicular to both the field and the velocity (i.e. it uses the cross product). The magnetic force can not change a particle's speed or KE**

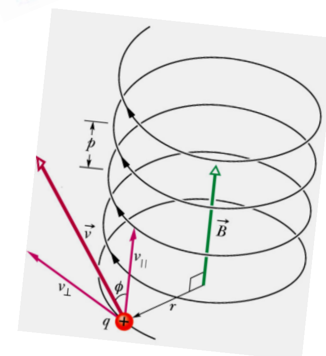
$$\vec{F}_B = q\,\vec{v} \times \vec{B}$$



- **A charged particle moving in a uniform magnetic field moves in a circle or a spiral.**

$$R = \frac{mv}{qB} \qquad \omega_c = \frac{2\pi}{\tau_c} = \frac{qB}{m}$$





- **Because currents are moving charges, a wire carrying current in a magnetic field feels a force also using cross product. This force is responsible for the motor effect.**

$$\vec{F}_B = i\vec{L} \times \vec{B}$$

- **For a current loop, the Magnetic dipole moment, torque, and potential energy are given by:**

$$\vec{\mu} \equiv N\,i\,A\,\hat{n}$$

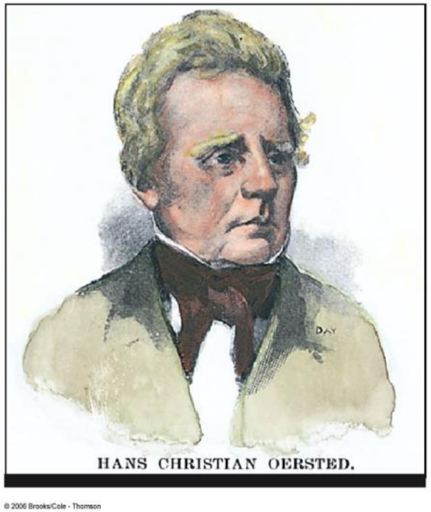
$$\vec{\tau}_m = \vec{\mu} \times \vec{B}$$

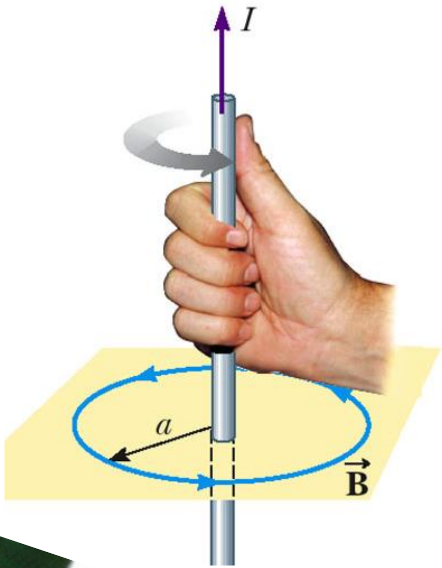
$$U_m = -\vec{\mu} \circ \vec{B}$$

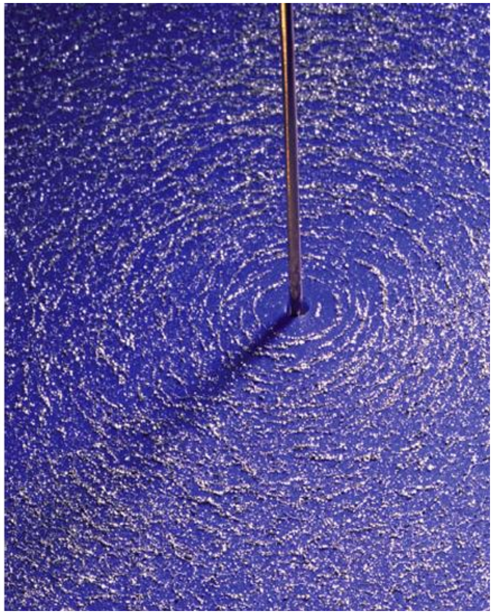
Magnetic fields are due to currents

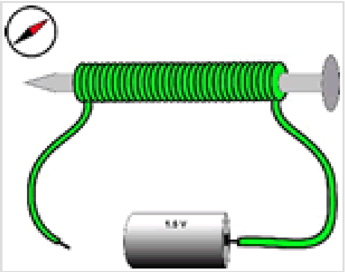
Oersted - 1820: A magnetic compass is deflected by current

→ Magnetic fields are due to currents (free charges & in wires)







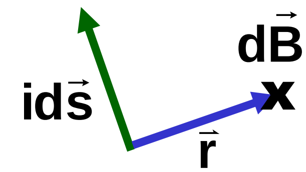


In fact, currents are the only way to create magnetic fields.

The magnitude of the field created is proportional to $i\,\Delta s$ (current - length)

The Biot-Savart Law (1820)

- Same basic role as Coulomb's Law: magnetic field due to a source
- Source strength measured by "current-length" i ds
- Falls off as inverse-square of distance
- New constant μ_0 measures "permeability"
- Direction of B field depends on a cross-product (Right Hand Rule)



Differential addition to field at P due to distant source ids

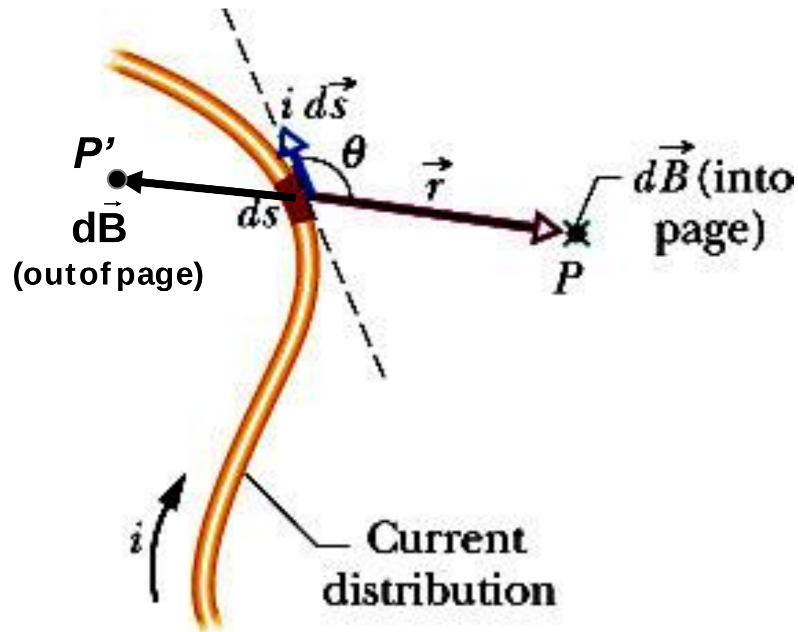
$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{id\vec{s}\times\hat{r}}{r^2}$$

Unit vector along r - from source to P

10^{-7} exactly

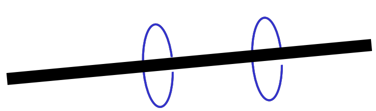
"vacuum permeability"

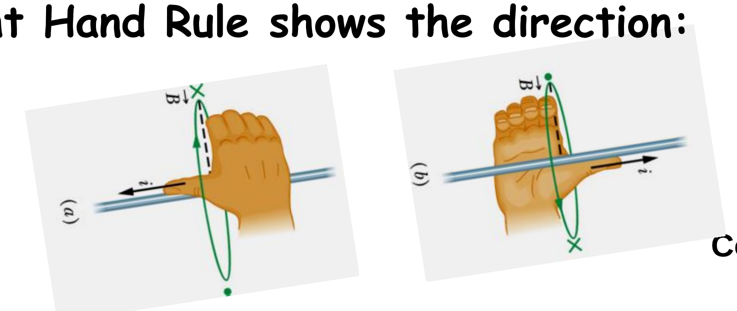
$\mu_0 = 4\pi \times 10^{-7}$ T.m/A.



Find **total** field B by integrating over the whole current region (need lots of symmetry)

For a straight wire the magnetic field lines are circles wrapped around it. Another Right Hand Rule shows the direction:

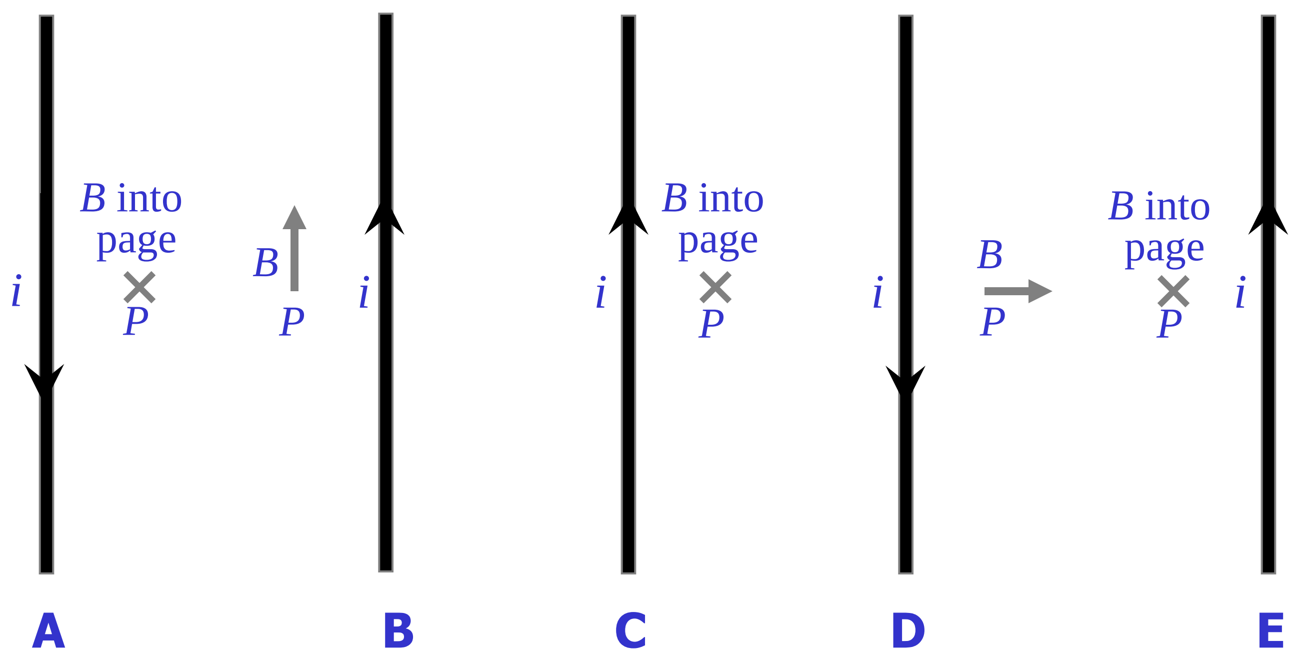




$$dB = \frac{\mu_0}{4\pi}\frac{ids\sin(\theta)}{r^2}$$

Direction of Magnetic Field

10 – 1: Which sketch below shows the correct direction of the magnetic field, B, near the point P?

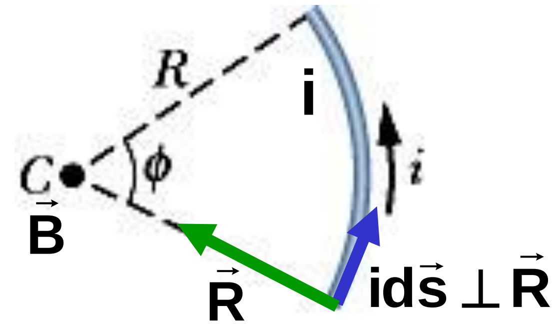


Use RH rule for current segments:
thumb along **ids** - curled fingers show B

Example: Magnetic field at the center of a current arc

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s}\times\hat{r}}{r^2}$$

- Circular arc carrying current, constant radius R
- Find <u>B</u> at center, point C
- ϕ is included arc angle, not the cross product angle
- Angle θ for the cross product is always 90^0
- dB at center C is up out of the paper
- ds = R$d\phi'$

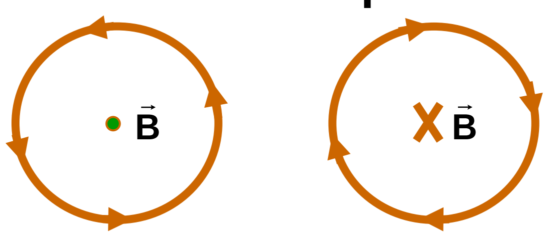


$$dB = \frac{\mu_0}{4\pi} i \frac{ds}{R^2} = \frac{\mu_0}{4\pi} i \frac{d\varphi'}{R}$$

- integrate on arc angle ϕ' from 0 to ϕ

$$B = \frac{\mu_0 i}{4\pi R}\int_0^{\phi} d\phi' = \frac{\mu_0 i}{4\pi R}\phi \qquad \phi \text{ in radians}$$

- For a circular loop of current - $\phi = 2\pi$ radians:



$$B = \frac{\mu_0 i}{2R} \quad \text{(loop)}$$

Right hand rule for wire segments

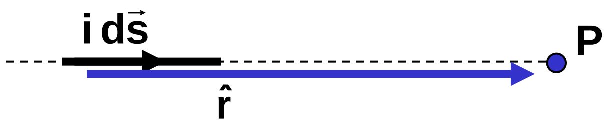
Thumb points along the current. Curled fingers show direction of B

Another Right Hand Rule (for loops):
Curl fingers along current, thumb shows direction of B at center

?? What would formula be for ϕ = 45°, 180°, 4π radians ??

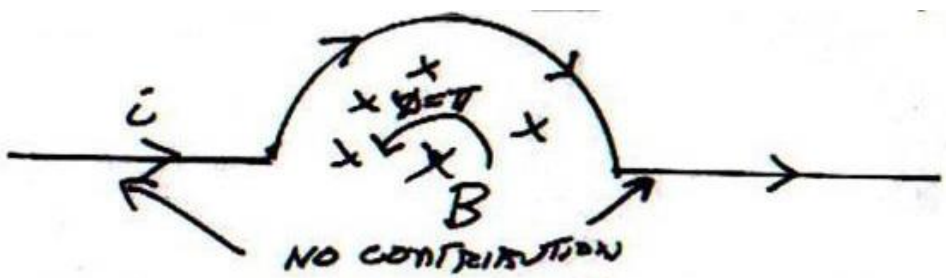
Examples:

FIND B FOR A POINT LINED UP WITH A SHORT STRAIGHT WIRE



$$i\,d\vec{s} \times \hat{r} = 0 \qquad dB = \frac{\mu_0}{4\pi r^2}\, i\, ds \sin(\theta) = 0$$

Find B AT CENTER OF A HALF LOOP, RADIUS = r

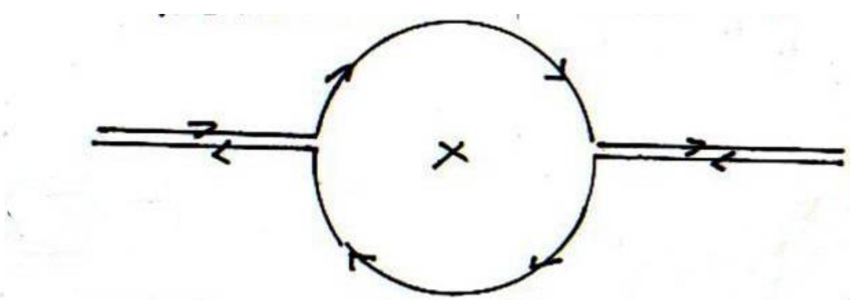


$$B = \frac{\mu_0 i}{4\pi r}\pi = \frac{\mu_0 i}{4r}$$

into page

Find B AT CENTER OF TWO HALF LOOPS

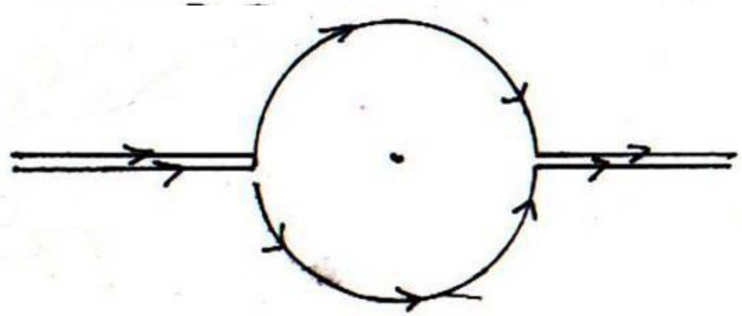
OPPOSITE CURRENTS



$$B = 2 \times \frac{\mu_0 i}{4\pi r}\pi = \frac{\mu_0 i}{2r}$$

same as closed loop

PARALLEL CURRENTS



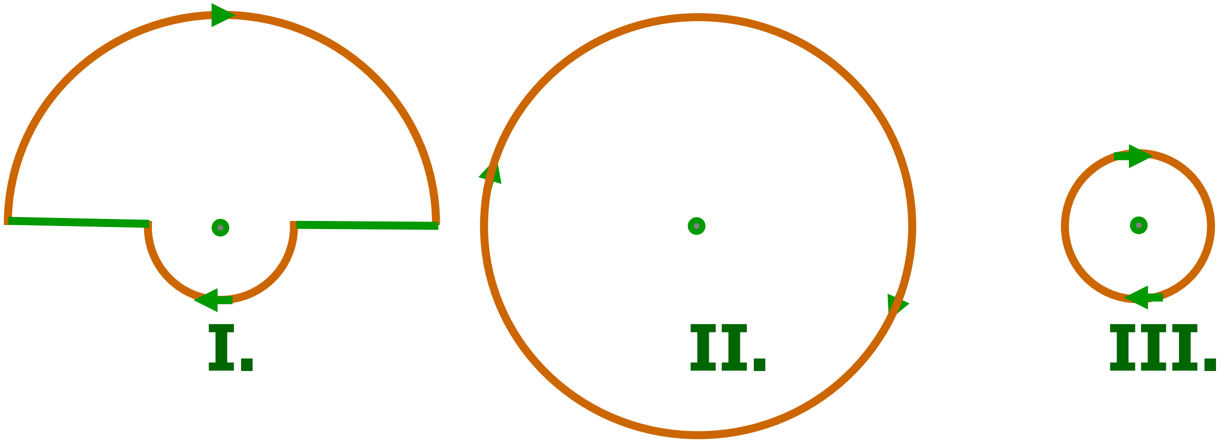
$$B = \frac{\mu_0 i}{4r} - \frac{\mu_0 i}{4r} = 0$$

into page   out of page

Magnetic Field from Loops

10 – 2: The three loops below have the same current. The smaller radius is half of the large one. Rank the loops by the magnitude of magnetic field at the center, greatest first.

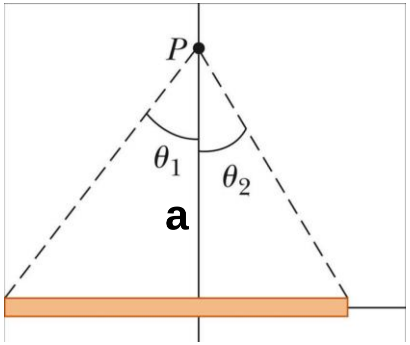
A. I, II, III.
B. III, I, II.
C. II, I, III.
D. III, II, I.
E. II, III, I.

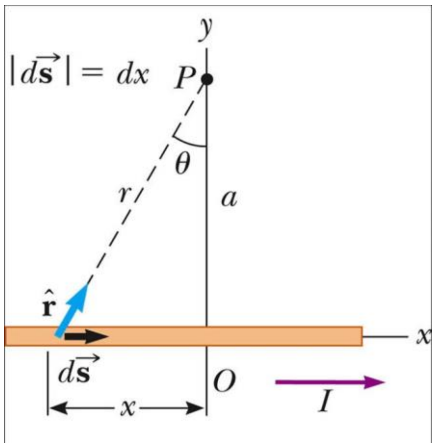


$$B = \frac{\mu_0 i}{4\pi R}\phi \qquad \phi \text{ in radians}$$

Hint: consider radius, direction, arc angle

Magnetic field due to current in a thin, straight wire





$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i d\vec{s}\times\vec{r}}{r^3}$$

- Current i flows to the right along x – axis
- Wire subtends angles θ_1 and θ_2
- Find <u>B</u> at point P, a distance a from wire.
- d<u>B</u> is out of page at P for <u>ds</u> anywhere along wire

Evaluate d<u>B</u> along d<u>s</u> using Biot Savart Law

- Magnitude of i.d<u>s</u> X <u>r</u> = i.r.dx.cos(θ).

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\cdot dx\cdot\cos(\theta)}{r^2}\hat{k}$$

- x negative as shown, θ positive, θ_1 positive, θ_2 negative

$$r = a/\cos(\theta) \qquad x = -a\tan(\theta) \qquad \frac{d}{d\theta}[\tan(\theta)] = \sec^2(\theta) = 1/\cos^2(\theta)$$

$$\therefore \quad dx = -a\cdot d\theta/\cos^2(\theta)$$

$$|d\vec{B}| = -\frac{\mu_0 i}{4\pi a}\cos(\theta)d\theta$$

Integrate on θ from θ_1 to θ_2:

$$B = \int_{\theta_1}^{\theta_2} dB = -\frac{\mu_0 i}{4\pi a}\int_{\theta_1}^{\theta_2}\cos(\theta)d\theta = \frac{\mu_0 i}{4\pi a}[\sin(\theta_1) - \sin(\theta_2)]$$

General result - applications follow

Magnetic field due to current in thin, straight wires

$$B = \frac{\mu_0 i}{4\pi a}[\sin(\theta_1) - \sin(\theta_2)]$$

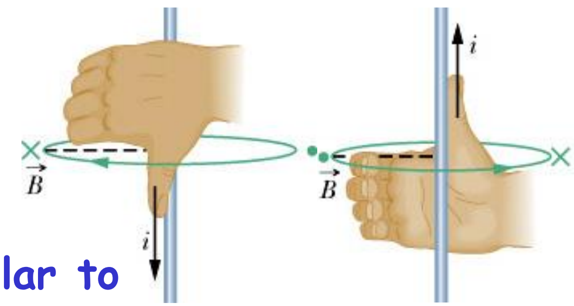
Example: Infinitely long, thin wire:

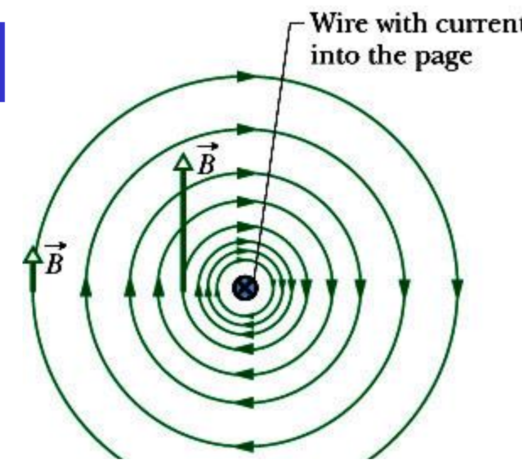
Set $\theta_1 = \pi/2,\ \theta_2 = -\pi/2$ [+ direction was CW in sketch]

$$B = \frac{\mu_0 i}{2\pi a}$$

a is distance perpendicular to wire through P

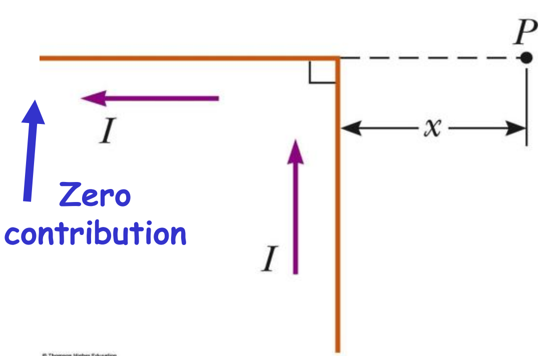
RIGHT HAND RULE FOR A WIRE





FIELD LINES ARE CIRCLES
THEY DO NOT BEGIN OR END

Example: Field at P due to Semi-Infinite wires:

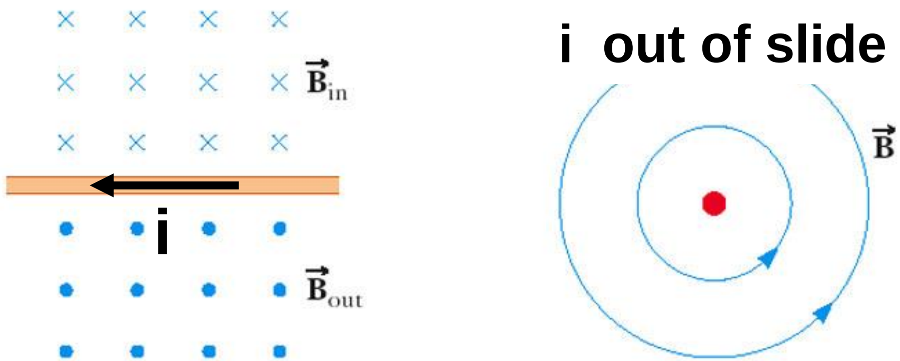


Set $\theta_1 = \pi/2,\ \theta_2 = 0$

$$|B| = \frac{\mu_0 i}{4\pi a}$$

Into slide at point P
Half the magnitude for a fully infinite wire

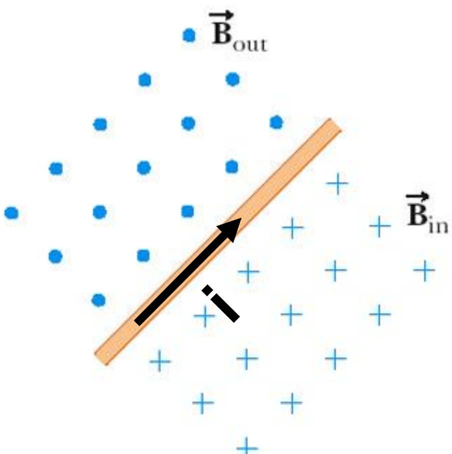
Magnetic Field lines near a straight wire carrying current

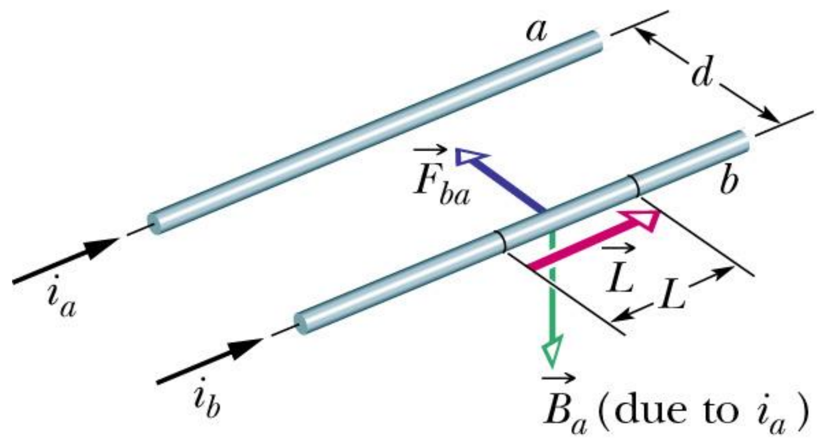


(a) (b)

When two parallel wires are carrying current, the magnetic field from one causes a force on the other.

$$\vec{F}_{a,b} = i_b \vec{L}_b \times \vec{B}_a$$

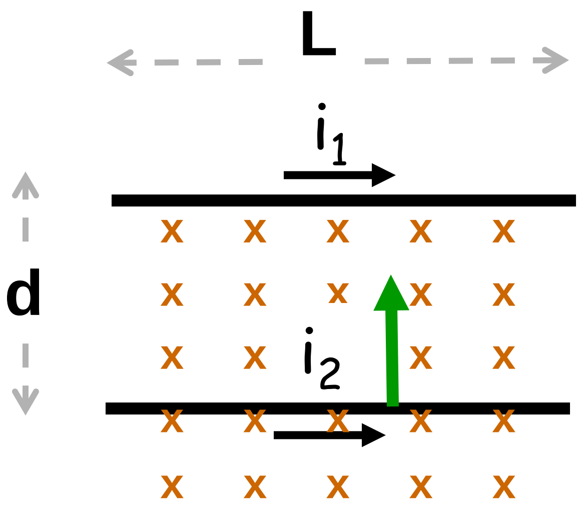


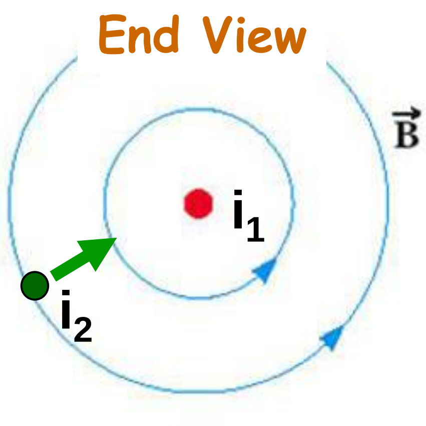


$$B_a = \frac{\mu_0 i_a}{2\pi R}$$

- The force is attractive when the currents are parallel.
- The force is repulsive when the currents are anti-parallel.

Magnitude of the force between two long parallel wires





- Third Law says: $F_{12} = -F_{21}$
- Use result for B due to infinitely long wire

$$B_1 = \frac{\mu_0 i_1}{2\pi d}$$

Due to 1 at wire 2
Into page via RH rule

- Evaluate F_{12} = force on 2 due to field of 1

$$\vec{F}_{12} = i_2 \vec{L}_2 \times \vec{B}_1$$

$$|\vec{F}_{1,2}| = i_2 L\, B_1$$

$i_2 L$ is normal to B
Force is toward wire1

$$\therefore \quad F_{1,2} = \frac{\mu_0}{2\pi} \frac{i_1 i_2}{d} L$$

$F_{21} = - F_{12}$

- Attractive force for parallel currents
- Repulsive force for opposed currents

Example: Two parallel wires are 1 cm apart $|i_1| = |1_2| = 100$ A.

$$F/L = \text{force per unit length} = \frac{2\text{x}10^{-7}\text{ x } 100 \text{ x } 100}{.01} = 0.2 \text{ N/m}$$

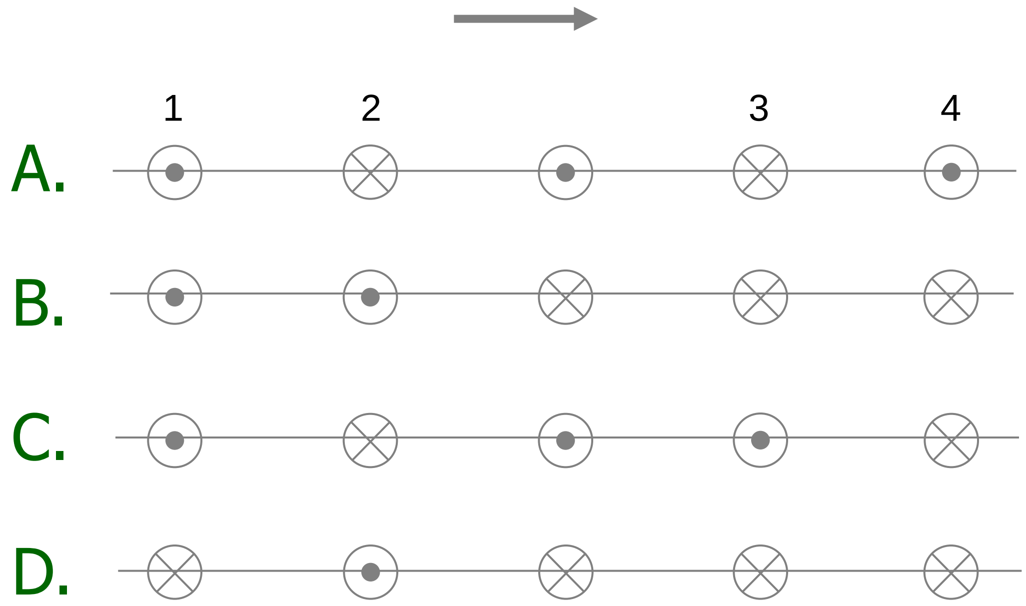
$$F = 0.2 \text{ N} \quad \text{for } L = 1 \text{ m}$$

Forces on parallel wires carrying currents

10 – 3: Which of the four situations below results in the greatest force to the right on the central conductor? The currents in all the wires have the same magnitude.

$$B = \frac{\mu_0 i}{2\pi R} \qquad \vec{F}_{tot} = i\,\vec{L} \times \vec{B}_{tot}$$

greatest F ?



Hints: Which pairings with center wire are attractive and repulsive?
or
What is the field midway between wires with parallel currents?
What is the net field directions and relaative magnitudes at center wire

Ampere's Law

- Derivable from Biot-Savart Law
- Sometimes a way to find B, given the current that creates it
- But B is inside an integral → usable only for high symmetry (like Gauss' Law)

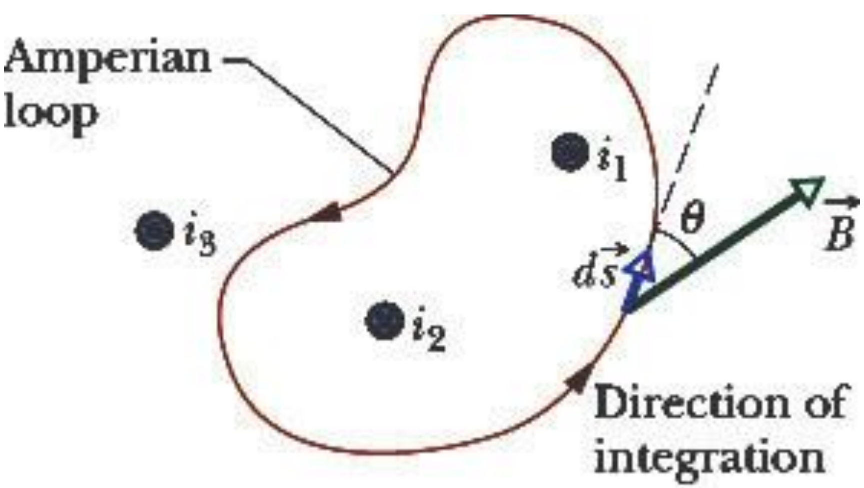
- An "Amperian loop" is a closed path of any shape
- Add up (integrate) components of B along the loop path.

$$\oint \vec{B} \circ d\vec{s} = \mu_0 i_{enc}$$

i_{enc}= net current passing through the loop

To find B, you have to be able to do the integral, then solve for B

Picture for applications:



- Only the tangential component of B along ds contributes to the dot product
- Current outside the loop (i_3) creates field but doesn't contribute to the path integral
- Another version of RH rule:
  - curl fingers along Amperian loop
  - thumb shows + direction for net current

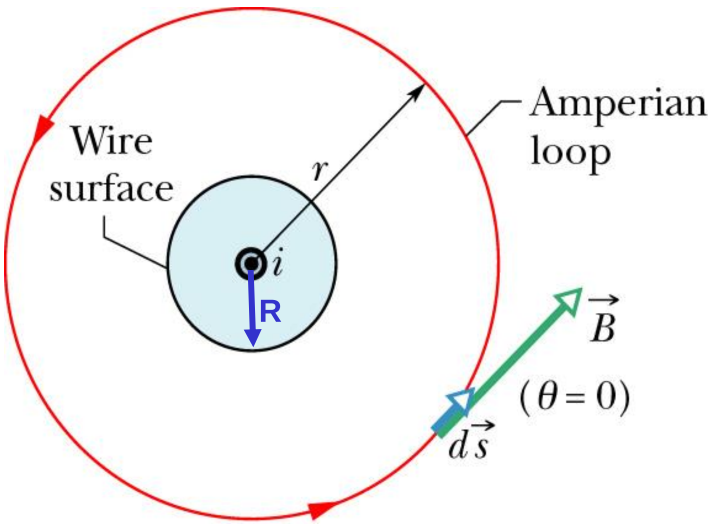
Example: Find magnetic field outside a long, straight, possibly fat, cylindrical wire carrying current

We used the Biot-Savart Law to show that for a thin wire

$$B = \frac{\mu_0 i}{2\pi r}$$

Now use Ampere's Law to show it again more simply and for a fat wire.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}$$



Amperian loop outside R can have any shape

Choose a circular loop (of radius r>R) because field lines are circular about a wire.

B and ds are then parallel, and B is constant everywhere on the Amperian path

$$\oint \vec{B} \circ d\vec{s} = B \times 2\pi r = \mu_0 i_{enc}$$

The integration was simple. i_{enc} is the total current.

Solve for B to get our earlier expression:

$$B = \frac{\mu_0 i}{2\pi r} \quad \text{outside wire}$$

R has no effect on the result.

Magnetic field inside a long straight wire carrying current, via Ampere's Law

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 i_{enc}$$

Assume current density J = i/A is uniform across the wire cross-section and is cylindrically symmetric.

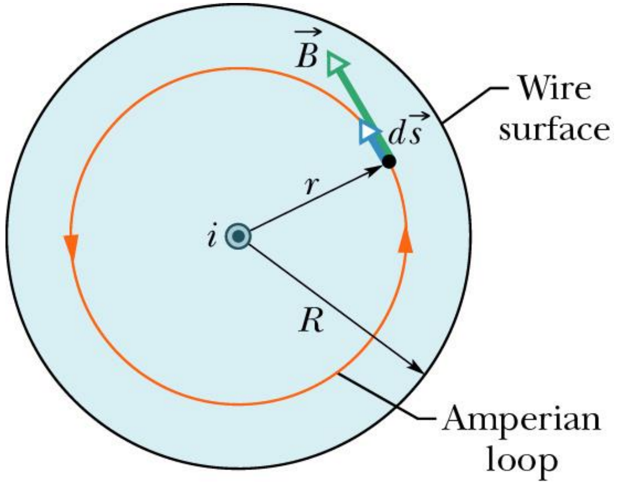
Field lines are again concentric circles

B is axially symmetric again

Again draw a circular Amperian loop around the axis, of radius r < R.

The enclosed current is less than the total current i, because some is outside the Amperian loop. The amount enclosed is

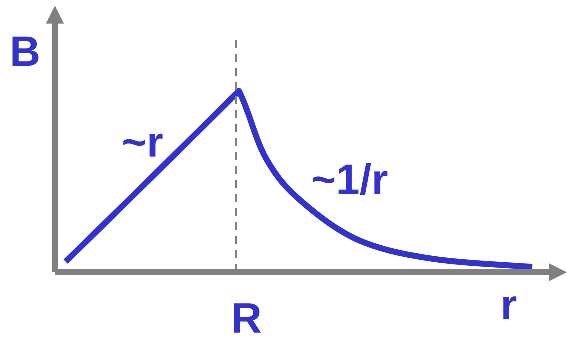
$$i_{enc} = i\frac{\pi r^2}{\pi R^2}$$



Apply Ampere's Law:

$$\oint \vec{B}\cdot d\vec{s} = B2\pi r = \mu_0 i_{enc} = \mu_0 i\frac{r^2}{R^2}$$

$$B = \left(\frac{\mu_0 i}{2\pi R}\right)\left(\frac{r}{R}\right) \quad r < R \quad \text{inside wire}$$

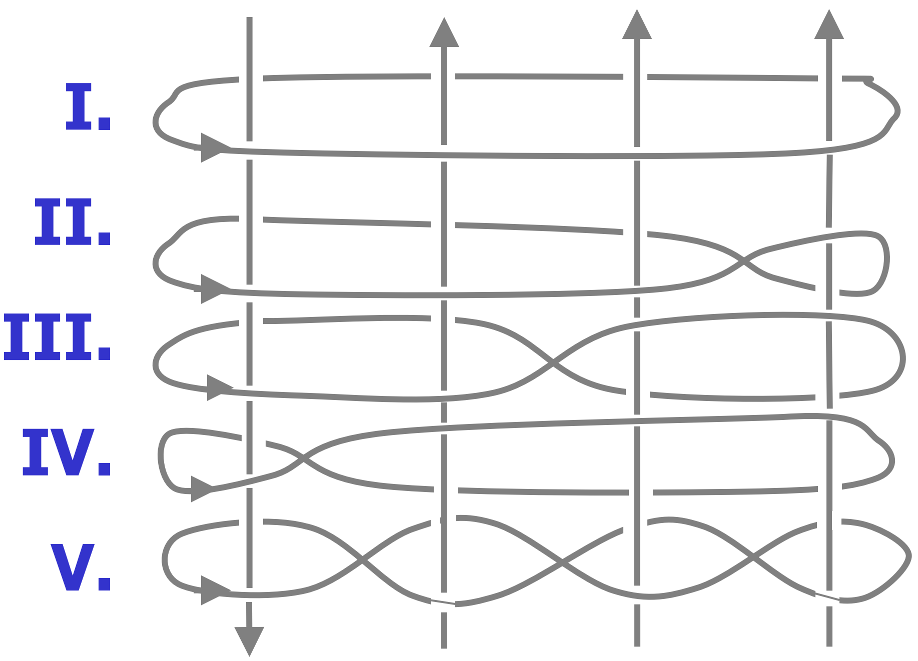


Outside (r>R), the wire looks like an infinitely thin wire (previous expression)
Inside: B grows linearly up to R

Counting the current enclosed by an Amperian Loop

**10 – 4: Rank the Amperian paths shown by the value of $\oint \vec{B} \cdot d\vec{s}$ along each path, taking direction into account and putting the most positive ahead of less positive values.
All of the wires are carrying the same current..**

A. I, II, III, IV, V.
B. II, III, IV, I, V.
C. III, V, IV, II, I.
D. IV, V, III, I, II.
E. I, II, III, V, IV.



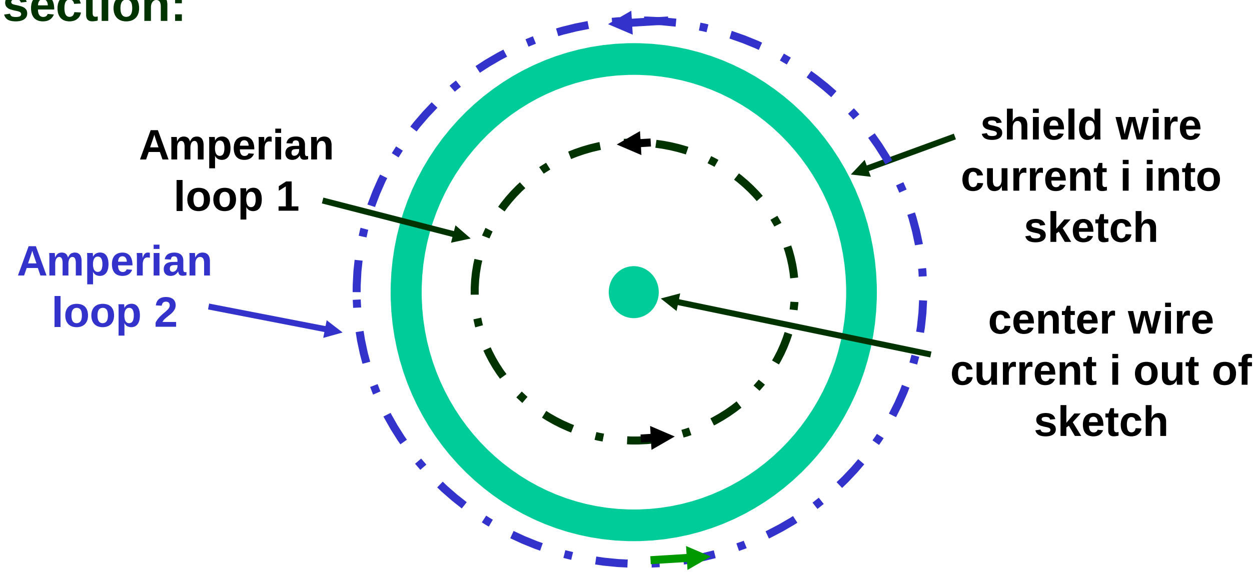
$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}$$

Another Ampere's Law example

Why use COAXIAL CABLE for CATV and other applications?

Find B inside and outside the cable

Cross section:



Inside – use Amperian loop 1:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i \quad = B x 2\pi r$$

$$B = \frac{\mu_0 i}{2\pi r}$$

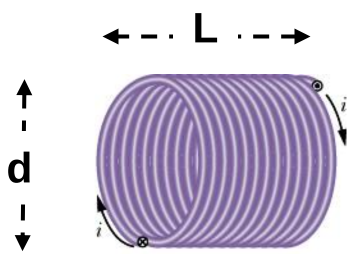
Outer shield does not affect field inside
Reminiscent of Gauss's Law

Outside – use Amperian loop 2:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} = 0$$

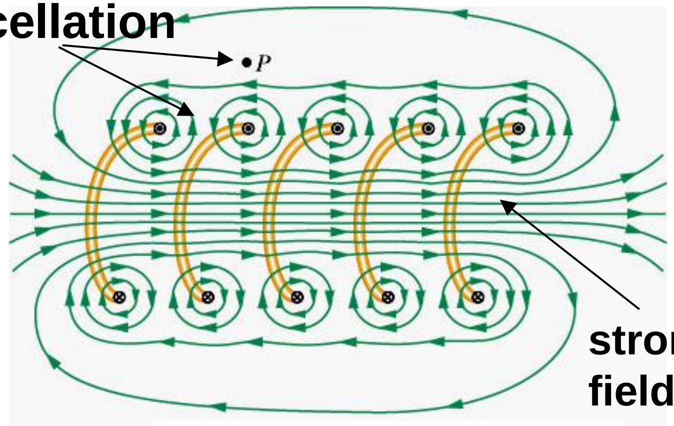
Zero field outside due to opposed currents + radial symmetry
Losses and interference suppressed

Solenoids strengthen fields by using many loops



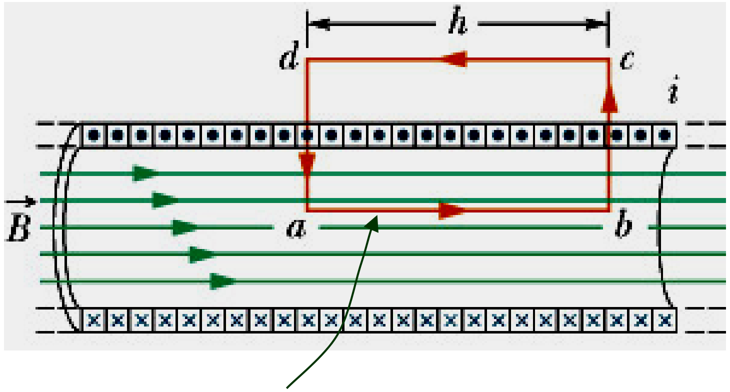
$n \equiv \#\,\text{coils}/\text{unit length} = N/L$

Approximation: field is constant inside and zero outside (just like capacitor)



"Long solenoid" → d << L

FIND FIELD INSIDE IDEAL SOLENOID USING AMPERIAN LOOP abcda



only section that has non-zero contribution

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}$$

$$\oint \vec{B} \cdot d\vec{s} = B_{inside} h = \mu_0 i_{enc} = \mu_0 inh$$

$$B = \mu_0 in$$

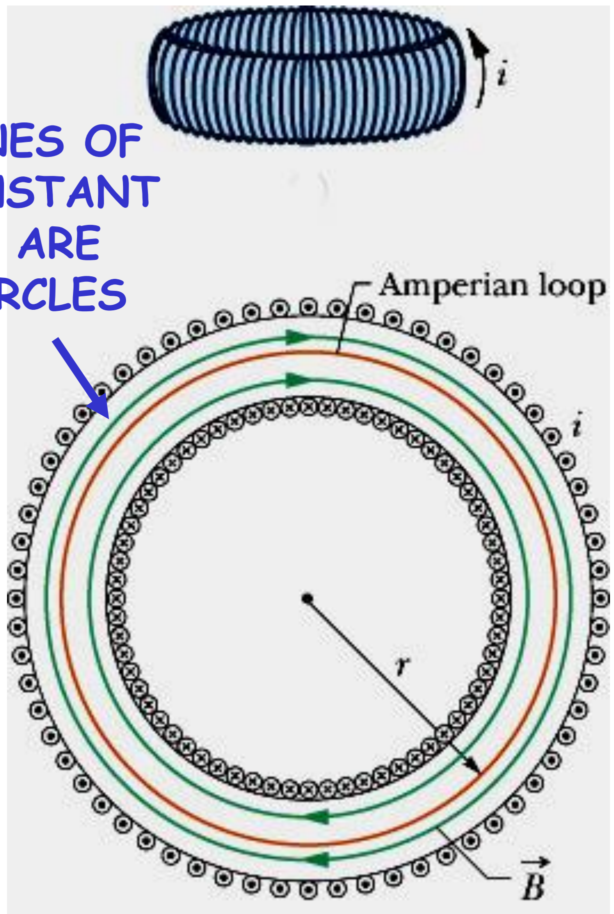
inside ideal solenoid

- Outside B = 0, no contribution from path c-d
- B is perpendicular to ds on paths a-d and b-c
- Inside B is uniform and parallel to ds on path a-b

Toroid: A long solenoid bent into a circle

i outside flows up

LINES OF CONSTANT B ARE CIRCLES



AMPERIAN LOOP IS A CIRCLE ALONG <u>B</u>

Find the magnitude of B field inside

- Draw an Amperian loop parallel to the field, with radius *r* (inside the toroid)
- The toroid has a total of *N* turns
- The Amperian loop encloses current *Ni*.
- B is constant on the Amperian path.

$$\oint \vec{B}\cdot d\vec{s} = B \times 2\pi r = \mu_0 i_{enc} = \mu_0 iN$$

$$B = \frac{\mu_0 iN}{2\pi r} \quad \text{inside toroid}$$

- N times the result for a long thin wire
- Depends on r
- Also same result as for long solenoid

$$n \equiv \frac{N}{2\pi r} \ \text{(turns/unit length)} \Rightarrow B = \mu_0 in$$

Find B field outside

Answer $B = 0$ outside

Find B at point P on z-axis of a dipole (current loop)

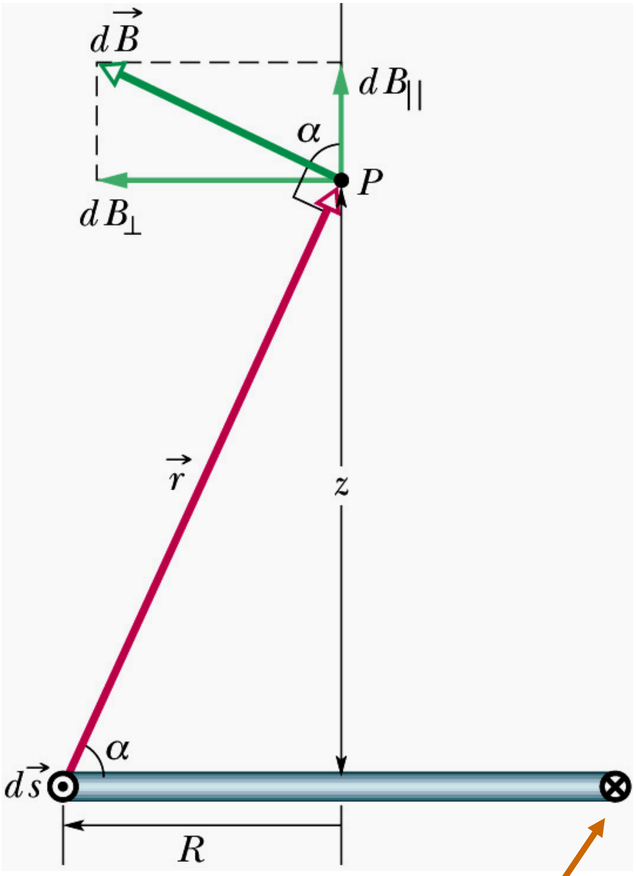
- We use the Biot-Savart Law directly

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\, d\vec{s} \times \hat{r}}{r^2} \qquad r = \sqrt{R^2 + z^2} \qquad \cos\alpha = \frac{R}{r}$$

$dB_\perp$ cancels by symmetry (normal to z - axis)

$$dB_z = dB_{||} = dB\cos(\alpha) = \frac{\mu_0}{4\pi}\frac{i\, ds \cos(\alpha)}{R^2 + z^2}$$

$$dB_z = \frac{\mu_0}{4\pi}\frac{iR}{(R^2 + z^2)^{3/2}} ds \qquad ds = R\, d\phi$$



- Integrate around the current loop on ϕ – the angle at the center of the loop.
- The field is perpendicular to r but by symmetry the part of B normal to z-axis cancels around the loop - only the part parallel to the z-axis survives.

i is into page

$$B_z = \int dB_z = \frac{\mu_0}{4\pi}\frac{iR}{(R^2 + z^2)^{3/2}}\int ds = \frac{\mu_0}{4\pi}\frac{iR^2}{(R^2 + z^2)^{3/2}}\int_0^{2\pi} d\phi$$

$$B(z) = \frac{\mu_0 i\, \pi R^2}{2\pi(R^2 + z^2)^{3/2}}$$

as before

$$B(z = 0) = \frac{\mu_0 i}{2R}$$

recall definition of Dipole moment

$$\mu \equiv NiA = i\pi R^2$$

B field on the axis of a dipole (current loop), continued

Far, far away: suppose z >> R

$$B(z) = \frac{\mu_0 i \pi R^2}{2\pi(R^2 + z^2)^{3/2}} \rightarrow \frac{\mu_0 i \pi R^2}{2\pi z^3}$$

Same $1/z^3$ dependence as for electrostatic dipole

Dipole moment vector $\vec{\mu}$ is normal to loop (RH Rule).

$\vec{\mu} \equiv NiA\hat{\mu}$

$N \equiv$ number of turns $= 1$

$A \equiv$ area of loop $= \pi R^2$

$\Rightarrow \pi R^2 i = |\mu|$ above

For any current loop, along z axis with |z| >> R

$$\therefore \vec{B}(z) \approx \frac{\mu_0 \vec{\mu}}{2\pi z^3}$$

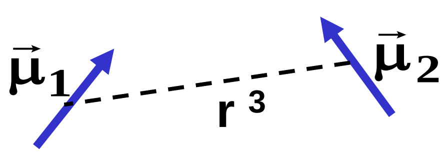
For charge dipole

$$\vec{E}(z) \approx \frac{1}{2\pi\varepsilon_0} \frac{\vec{p}}{z^3}$$

Current loops are the elementary sources of magnetic field:

- Creates dipole fields with source strength $\vec{\mu}$
- Dipole feels torque to another $\vec{\mu}$ in external B field $\vec{\tau} = \vec{\mu} \times \vec{B}$

Dipole-dipole interaction:

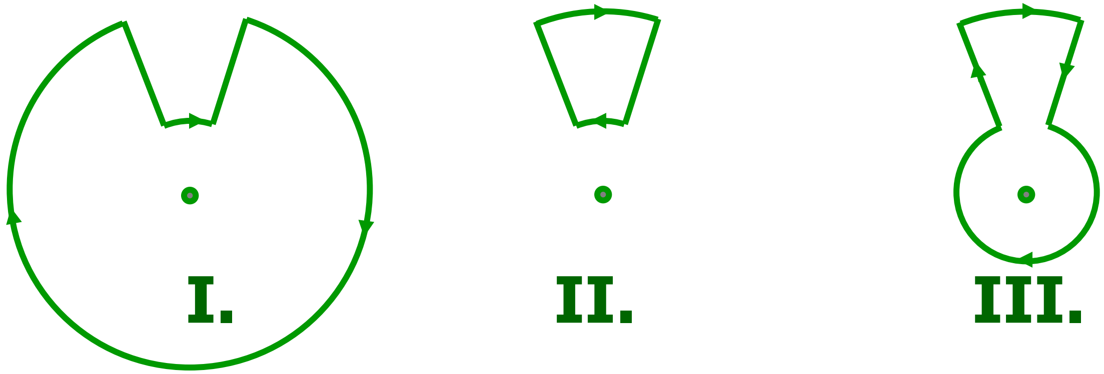


Torque depends on $\vec{\mu}_1 \times \vec{\mu}_2$

Try this at home

10-5: The three loops below have the same current. Rank them in terms of the magnitude of magnetic field at the point shown, greatest first.

A. I, II, III.
B. III, I, II.
C. II, I, III.
D. III, II, I.
E. II, III, I.



Hint: consider radius, direction, arc angle

$$B = \frac{\mu_0 i}{4\pi R}\phi \qquad \phi \text{ in radians}$$

Answer: B

Summary: Lecture 10 Chapter 29 – Magnetic Fields from Currents

Biot Savart Law

Source: current-length $i\,d\vec{s}$

$$d\vec{B} = \frac{\mu_0}{4\pi}\,\frac{i\,d\vec{s} \times \hat{r}}{r^2}$$

μ_0 = Permeability $= 4\pi \times 10^{-7}$ $\frac{\text{Tesla-m}}{\text{A}}$

$\hat{r}$ = unit vector along $\vec{r}$

$i\,d\vec{s} \times \hat{r} \rightarrow i\,ds \sin\theta$ (scalar form)

$\hat{r}$ from source to p

- Right Hand Rule: Thumb along $i\,d\vec{s}$, Curled fingers show $d\vec{B}$

Method for finding $\vec{B}$ at point P: integrate over source, vary $\vec{r}$, p is fixed.

Infinite straight wire

$$B = \frac{\mu_0 i}{2\pi r}$$

Point on axis of straight wire

$i\,d\vec{s} \times \hat{r} = 0$

$dB = 0$.

Arc of current – B at center

$$B = \frac{\mu_0 i}{4\pi r}\,\varphi$$

ϕ in radians

Full loop ($\phi = 2\pi$)

$$B = \frac{\mu_0 i}{2r}$$

RH Rule for loops. Finger curl, thumb

B on axis of current loop

Elementary dipole

$\vec{\mu}$ = Dipole moment $= NiA\,\hat{n}$

$$\vec{B}(z) = \frac{\mu_0\,\vec{\mu}}{2\pi (z^2 + R^2)^{3/2}} \rightarrow \frac{\mu_0\,\vec{\mu}}{2\pi z^3}$$

Force between 2 straight wires

$$\frac{F}{L} = \frac{\mu_0\, i_1 i_2}{2\pi d}$$

Attractive: parallel i's
Repulsive: opposite i's

Amperes Law (limited to symmetry cases)

$$\oint \vec{B} \cdot d\vec{s} = \mu_0\, i_{enclosed}$$

Amperian loop

$B = \frac{\mu_0 i}{2\pi r}$ (long wire) outside fat wire

$B = \frac{\mu_0 i r}{2\pi R^2}$ at $r < R$ inside fat wire

$B = \frac{\mu_0 i}{2\pi r}$ inside coax

i up, i down

$B = 0$ outside coax

Ideal solenoid

$d \ll L$

n = #turns/unit length

$B_{inside} = \mu_0 i n$ uniform

$B_{outside} = 0$

Toroid:

$$B = \mu_0 i n = \frac{\mu_0 i N}{2\pi r}$$

Thin wire, asymmetric point

$$B = \frac{\mu_0 I}{4\pi a}\left[\sin(\theta_1) - \sin(\theta_2)\right]$$